# A NEW GEOMETRY FOR SCHOOLS

CLEMENT V. DURELL

*STAGE B*

PARTS 1, 2 & 3

# A NEW GEOMETRY

# A NEW GEOMETRY
## FOR SCHOOLS

BY

CLEMENT V. DURELL, M.A.

AUTHOR OF "GENERAL ARITHMETIC," "A NEW ALGEBRA FOR SCHOOLS," "MATRICULATION TRIGONOMETRY," ETC.

LONDON

G. BELL AND SONS, LTD

1960

A volume of HINTS AND SOLUTIONS is issued for the use of teachers

*First published* 1939
*Reprinted* 1939 (*twice*), 1940 (*twice*), 1941, 1942 (*twice*), 1943, 1944, 1945, 1946, 1947, 1948, 1951, 1952, 1954, 1955, 1956, 1957, 1958, 1960

Printed in Great Britain by
NEILL & CO. LTD., EDINBURGH

# PREFACE

It is now almost fourteen years since the author's *Elementary Geometry* was published, and, in writing this entirely new book, he has taken the opportunity to recast his treatment of the subject in the light of the experience gained, and the suggestions received, since *Elementary Geometry* appeared. He has been able, also, as will be seen later, to make full use of the Second Report of the Mathematical Association on the Teaching of Geometry.

This book contains a course of geometry from the first stage up to the standard of the School Certificate and similar examinations. The object of the author has been to provide a treatment which lends itself both to class-teaching and to individual use by the pupil. The plan adopted throughout is to develop each group of geometrical facts by the following successive stages:—

(i) *Examples for oral discussion.*

These are illustrated extensively by diagrams in order to simplify black-board work.

This oral work gives the pupil a clear understanding of the relevant facts, familiarises him with the arguments which will be used later in the formal proofs of theorems, and trains him in methods for solving riders. It includes, when appropriate, questions in which the data are numerical.

(ii) *An exercise of numerical examples.*

This gives practice in applying the facts deduced from oral discussion and ensures a firm grasp of these facts.

(iii) *Formal proofs of the corresponding theorems.*

The preliminary work makes it possible to deal with these proofs rapidly. Practice in writing out theorems is essential for examination purposes, but it will often be found sufficient

to confine this to the key-theorem of each group, regarding the others as simple riders.

(iv) *An exercise of riders.*

The early examples in each exercise are direct and very simple applications of the properties of the group. Some assistance is supplied for the harder examples, but notes on method and hints of useful constructions are included in the text.

The prominence in the text of the examples for oral discussion is due to the author's conviction that this not only facilitates the learning of formal proofs of theorems but, what is far more important, is the best method of strengthening the power of the pupil to tackle riders by showing him the types of constructions most often required, by helping him to assimilate the fundamental facts, and by making him familiar with the forms of argument he must be able to employ. Although these examples are called "oral," it is suggested that all pupils should be required to *write down* the answers to each question.

The examples in each exercise are classified under three heads:

(A) *Normal course:* plain numbers.

These examples cover all essential types and have been graded carefully. Most of them should be done by all pupils.

(B) *Extra practice:* numbers enclosed in brackets.

These examples provide further training if needed and are parallel to those in A and do *not* extend the ground covered.

(C) *Advanced course:* asterisked numbers.

These are intended only for those pupils who run ahead of the class.

For the convenience of teachers who prefer to make their own selection, these groups are not printed in separate sections, but the examples are arranged in order of difficulty.

Other features of the book which may be mentioned are:

(i) The extensive use of diagrams in the exercises.
This diminishes the difficulties of the pupil and gives further opportunity for oral work.

(ii) The expression of geometrical facts in trigonometrical form when this alternative is instructive, *e.g.* in connection with areas, the extensions of Pythagoras' theorem, etc.

(iii) Applications to solid geometry throughout the course.

(iv) A fuller treatment of loci than is customary.

The use of the small-letter notation for angles recommended in the Second Report on the Teaching of Geometry has been adopted except in such cases as the use is liable to cause confusion; and advantage has been taken of many other instructive suggestions made in that report.

The work in Stage A gives practice in the use of instruments, and deals with the fundamental facts associated with parallels, congruence, and similarity. A much longer introductory course in which Stage A methods are extended to the subject-matter of the whole course of school certificate geometry is provided by *Simplified Geometry*, Durell and Tuckey, which may be used in place of the Stage A section of this book.

An alternative arrangement of the material contained in *A New Geometry* is issued, under the title **Exercises and Theorems in Geometry.** This is intended to meet the requirements of those teachers who prefer an arrangement similar to that which the author adopted in his *Concise Geometry.* In the alternative book the Stage A geometry is identical with that in *A New Geometry*, but thereafter Exercises, Constructions and Theorems respectively are, for the most part, collected together in separate sections of the book.

A volume of **Hints and Solutions** will be available for use with either book.

The author owes especial gratitude to Mr K. R. Imeson for helpful advice. He tenders also cordial thanks to the several teachers who have been good enough, out of their experience, to make valuable suggestions and criticisms, which have enabled him to take into due consideration different needs and varying points of view.

C. V. D.

*April*, 1939.

# CONTENTS

*NOTE This Geometry is issued complete and in parts. Full details of styles will be found facing the title page. The following is the Table of Contents of the whole book.*



## *STAGE A GEOMETRY*

# *STAGE B GEOMETRY*

## PART I



## PART II

### *SECTION* 1. *AREAS*

## *SECTION 2. THE CIRCLE*

## PART III

### *SIMILARITY*



### *APPENDIX*



### *ANSWERS*

## Symbols and Abbreviations

| | | | |
|---|---|---|---|
| $\therefore$ | therefore | adj. | adjacent |
| $\because$ | because | alt. | alternate |
| $=$ | is equal to | corr. | corresponding |
| $=$ | is equivalent to | ext. | exterior |
| $\equiv$ | is congruent to | int. | interior |
| $\neq$ | is not equal to | opp. | opposite |
| $\simeq$ | is approximately equal to | quad. | quadrilateral |
| | | rect. | rectangle |
| $\sim$ | the difference between | seg. | segment |
| $>$ | is greater than | sq. | square |
| $<$ | is less than | st. | straight |
| $\parallel$ | is parallel to | vert. opp. | vertically opposite |
| $\parallel$gram | parallelogram | | |
| $\angle$ | angle | | |
| rt. $\angle$ | right angle | | |
| $\triangle$ | triangle | | |
| $\bigcirc^{ce}$ | circumference | | |

[TABLES OVERLEAF

LOGARITHMS ADD

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 000 | 041 | 079 | 114 | 146 | | | | | | 4 | 7 | 11 | 14 | 18 | 21 | 24 | 28 | [illegible] |
| | | | | | | 176 | 204 | 230 | 255 | 279 | 3 | 5 | 8 | 10 | 13 | 15 | 18 | 20 | [illegible] |
| 2 | 301 | 322 | 342 | 362 | 380 | | | | | | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 15 | [illegible] |
| | | | | | | 398 | 415 | 431 | 447 | 462 | 2 | 3 | 5 | 6 | 8 | 9 | 11 | 13 | [illegible] |
| 3 | 477 | 491 | 505 | 519 | 531 | 544 | 556 | 568 | 580 | 591 | 1 | 2 | 4 | 5 | 6 | 7 | 9 | 10 | [illegible] |
| 4 | 602 | 613 | 623 | 633 | 643 | 653 | 663 | 672 | 681 | 690 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | [illegible] |
| 5 | 699 | 708 | 716 | 724 | 732 | 740 | 748 | 756 | 763 | 771 | 1 | 2 | 2 | 3 | 4 | 5 | 6 | 6 | [illegible] |
| 6 | 778 | 785 | 792 | 799 | 806 | 813 | 820 | 826 | 833 | 839 | 1 | 1 | 2 | 3 | 3 | 4 | 5 | 5 | [illegible] |
| 7 | 845 | 851 | 857 | 863 | 869 | 875 | 881 | 886 | 892 | 898 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 5 | [illegible] |
| 8 | 903 | 908 | 914 | 919 | 924 | 929 | 934 | 940 | 944 | 949 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | [illegible] |
| 9 | 954 | 959 | 964 | 968 | 973 | 978 | 982 | 987 | 991 | 996 | 1 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | [illegible] |

SQUARE ROOTS 1 TO 10 ADD

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 100 | 105 | 110 | 114 | 118 | 122 | 126 | 130 | 134 | 138 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 3 | [illegible] |
| 2 | 141 | 145 | 148 | 152 | 155 | 158 | 161 | 164 | 167 | 170 | 0 | 1 | 1 | 1 | 2 | 2 | 2 | 3 | [illegible] |
| 3 | 173 | 176 | 179 | 182 | 184 | 187 | 190 | 192 | 195 | 197 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | [illegible] |
| 4 | 200 | 202 | 205 | 207 | 210 | 212 | 214 | 217 | 219 | 221 | 0 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | [illegible] |
| 5 | 224 | 226 | 228 | 230 | 232 | 235 | 237 | 239 | 241 | 243 | 0 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | [illegible] |
| 6 | 245 | 247 | 249 | 251 | 253 | 255 | 257 | 259 | 261 | 263 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 2 | [illegible] |
| 7 | 265 | 266 | 268 | 270 | 272 | 274 | 276 | 277 | 279 | 281 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | [illegible] |
| 8 | 283 | 285 | 286 | 288 | 290 | 292 | 293 | 295 | 297 | 298 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | [illegible] |
| 9 | 300 | 302 | 303 | 305 | 307 | 308 | 310 | 311 | 313 | 315 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | [illegible] |

SQUARE ROOTS 10 TO 100 ADD

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 316 | 332 | 346 | 361 | 374 | 387 | 400 | 412 | 424 | 436 | 1 | 3 | 4 | 5 | 7 | 8 | 9 | 10 | 12 |
| 2 | 447 | 458 | 469 | 480 | 490 | 500 | 510 | 520 | 529 | 539 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 3 | 548 | 557 | 566 | 574 | 583 | 592 | 600 | 608 | 616 | 624 | 1 | 2 | 3 | 3 | 4 | 5 | 6 | 7 | 8 |
| 4 | 632 | 640 | 648 | 656 | 663 | 671 | 678 | 686 | 693 | 700 | 1 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 7 |
| 5 | 707 | 714 | 721 | 728 | 735 | 742 | 748 | 755 | 762 | 768 | 1 | 1 | 2 | 3 | 3 | 4 | 5 | 5 | 6 |
| 6 | 775 | 781 | 787 | 794 | 800 | 806 | 812 | 819 | 825 | 831 | 1 | 1 | 2 | 2 | 3 | 4 | 4 | 5 | 6 |
| 7 | 837 | 843 | 849 | 854 | 860 | 866 | 872 | 877 | 883 | 889 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 5 | 5 |
| 8 | 894 | 900 | 906 | 911 | 917 | 922 | 927 | 933 | 938 | 943 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| 9 | 949 | 954 | 959 | 964 | 970 | 975 | 980 | 985 | 990 | 995 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |

NATURAL SINES ADD

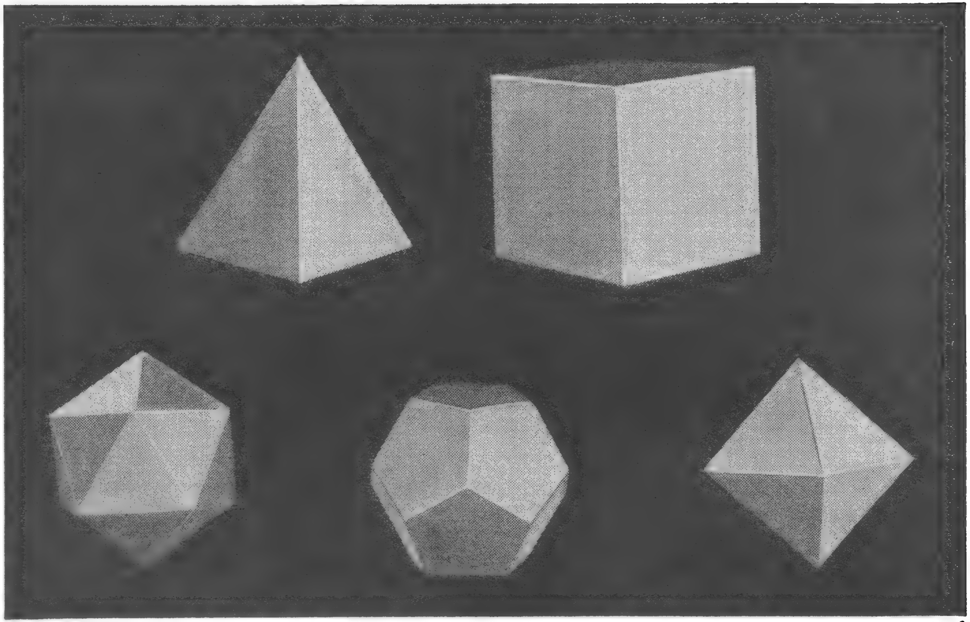
| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 6′ ·1 | 12′ ·2 | 18′ ·3 | 24′ ·4 | 30′ ·5 | 36′ ·6 | 42′ ·7 | 48′ ·8 | 54′ ·9 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | ·000 | 017 | 035 | 052 | 070 | 087 | 105 | 122 | 139 | 156 | 2 | 3 | 5 | 7 | 9 | 10 | 12 | 14 | 16 |
| | ·174 | 191 | 208 | 225 | 242 | 259 | 276 | 292 | 309 | 326 | 2 | 3 | 5 | 7 | 8 | 10 | 12 | 13 | 15 |
| | ·342 | 358 | 375 | 391 | 407 | 423 | 438 | 454 | 469 | 485 | 2 | 3 | 5 | 6 | 8 | 9 | 11 | 13 | 14 |
| | ·500 | 515 | 530 | 545 | 559 | 574 | 588 | 602 | 616 | 629 | 1 | 3 | 4 | 6 | 7 | 9 | 10 | 11 | 13 |
| | ·643 | 656 | 669 | 682 | 695 | 707 | 719 | 731 | 743 | 755 | 1 | 2 | 4 | 5 | 6 | 7 | 9 | 10 | 11 |
| | ·766 | 777 | 788 | 799 | 809 | 819 | 829 | 839 | 848 | 857 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| | ·866 | 875 | 883 | 891 | 899 | 906 | 914 | 921 | 927 | 934 | 1 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 6 |
| | ·940 | 946 | 951 | 956 | 961 | 966 | 970 | 974 | 978 | 982 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 3 | 4 |
| | ·985 | 988 | 990 | 993 | 995 | 996 | 998 | 999 | 999 | 1·00 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 |

NATURAL COSINES SUBTRACT

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 6′ ·1 | 12′ ·2 | 18′ ·3 | 24′ ·4 | 30′ ·5 | 36′ ·6 | 42′ ·7 | 48′ ·8 | 54′ ·9 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 1·00 | 1·00 | ·999 | ·999 | ·998 | ·996 | ·995 | ·993 | ·990 | ·988 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 |
| | ·985 | 982 | 978 | 974 | 970 | 966 | 961 | 956 | 951 | 946 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 3 | 4 |
| | ·940 | 934 | 927 | 921 | 914 | 906 | 899 | 891 | 883 | 875 | 1 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 6 |
| | ·866 | 857 | 848 | 839 | 829 | 819 | 809 | 799 | 788 | 777 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| | ·766 | 755 | 743 | 731 | 719 | 707 | 695 | 682 | 669 | 656 | 1 | 2 | 4 | 5 | 6 | 7 | 9 | 10 | 11 |
| | ·643 | 629 | 616 | 602 | 588 | 574 | 559 | 545 | 530 | 515 | 1 | 3 | 4 | 6 | 7 | 9 | 10 | 11 | 13 |
| | ·500 | 485 | 469 | 454 | 438 | 423 | 407 | 391 | 375 | 358 | 2 | 3 | 5 | 6 | 8 | 9 | 11 | 13 | 14 |
| | ·342 | 326 | 309 | 292 | 276 | 259 | 242 | 225 | 208 | 191 | 2 | 3 | 5 | 7 | 8 | 10 | 12 | 13 | 15 |
| | ·174 | 156 | 139 | 122 | 105 | 087 | 070 | 052 | 035 | 017 | 2 | 3 | 5 | 7 | 9 | 10 | 12 | 14 | 16 |

NATURAL TANGENTS ADD

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 6′ ·1 | 12′ ·2 | 18′ ·3 | 24′ ·4 | 30′ ·5 | 36′ ·6 | 42′ ·7 | 48′ ·8 | 54′ ·9 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | ·000 | 017 | 035 | 052 | 070 | 087 | 105 | 123 | 141 | 158 | 2 | 4 | 5 | 7 | 9 | 11 | 12 | 14 | 16 |
| | ·176 | 194 | 213 | 231 | 249 | 268 | 287 | 306 | 325 | 344 | 2 | 4 | 6 | 8 | 9 | 11 | 13 | 15 | 17 |
| | ·364 | 384 | 404 | 424 | 445 | 466 | 488 | 510 | 532 | 554 | 2 | 4 | 6 | 9 | 11 | 13 | 15 | 17 | 19 |
| | ·577 | 601 | 625 | 649 | 675 | 700 | 727 | 754 | 781 | 810 | 3 | 5 | 8 | 11 | 13 | 16 | 18 | 21 | 24 |
| | ·839 | 869 | 900 | 933 | 966 | | | | | | 3 | 6 | 9 | 13 | 16 | 19 | 22 | 26 | 29 |
| | | | | | | 1·000 | 1·036 | 1·072 | 1·111 | 1·150 | 4 | 7 | 11 | 15 | 19 | 23 | 27 | 30 | 34 |
| | 1·192 | 235 | 280 | 327 | 376 | | | | | | 5 | 9 | 14 | 19 | 23 | 28 | 33 | 38 | 42 |
| | | | | | | 428 | 483 | 540 | 600 | 664 | 6 | 12 | 18 | 24 | 30 | 36 | 42 | 48 | 54 |
| | 1·73 | 1·80 | 1·88 | 1·96 | 2·05 | 2·14 | 2·25 | 2·36 | 2·48 | 2·61 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| | 2·75 | 2·90 | 3·08 | 3·27 | 3·49 | | | | | | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 |
| | | | | | | 3·73 | 4·01 | 4·33 | 4·70 | 5·14 | – | – | – | – | – | – | – | – | – |
| | 5·67 | 6·31 | 7·12 | 8·14 | 9·51 | 11·4 | 14·3 | 19·1 | 28·6 | 57·3 | – | – | – | – | – | – | – | – | – |



W. Gadd

THE FIVE REGULAR SOLIDS

Tetrahedron Cube

Icosahedron Dodecahedron Octahedron

# STAGE B

## *PART I*

## PARALLELS, CONGRUENCE AND INEQUALITIES

The fundamental facts about

(i) the measurement of length and angle,

(ii) the relations between angles formed by a transversal and parallel lines,

(iii) the tests for congruence and similarity

have been discussed and illustrated in Stage A. These facts form a group of assumptions on which the course of elementary geometry is based; it is therefore convenient to repeat here the statements made there and give additional examples of their use.

### Angles at a Point

**Definition.** If a straight line CD meets another straight line ACB so as to make the two adjacent angles equal, each angle is called a **right angle.**

It is *assumed* that all right angles are equal.

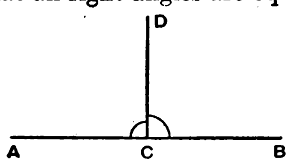


FIG. 203

CD is then said to be **at right angles** to ACB or **perpendicular** to ACB, and C is called the **foot of the perpendicular** from D to AB. **Never** speak of drawing a perpendicular from a line to a point.

If a straight line OA is perpendicular to each of two different straight lines OB and OC, it can be proved that OA is also perpendicular to every straight line through O in the plane OBC, and OA is then said to be **perpendicular to the plane** OBC.

## THEOREM 1

**If a straight line stands on another straight line, the sum of the adjacent angles so formed is equal to two right angles.**

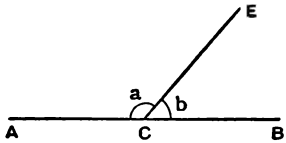



FIG. 204

With the notation of fig. 204, if ACB is a straight line,

then $\mathbf{a + b = 2}$ **rt. ∠s.**

*Abbreviation for reference:* adj. ∠s on st. line.

**Corollary. If any number of straight lines are drawn from a given point, the sum of the successive adjacent angles so formed is equal to four right angles.**

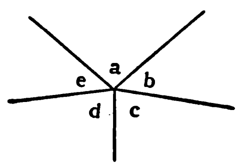



FIG. 205

With the notation of fig. 205,

$\mathbf{a + b + c + d + e = 4}$ **rt. ∠s.**

*Abbreviation for reference:* ∠s at a point.

*Formal proofs of Theorems* 1, 2 *are given in the Appendix, pp.* 540–1.

**Definition.** Two angles are called **supplementary** if their sum is equal to **two right angles** and either is called the **supplement** of the other.

In fig. 204, in which ACB is a straight line,

*a* and *b* are supplementary angles.

## THEOREM 2

**If the sum of two adjacent angles is equal to two right angles, the exterior arms of the angles are in the same straight line.**

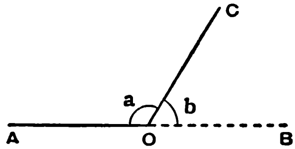



FIG. 206

With the notation of fig. 206,

if $\quad a + b = 2$ rt. $\angle$s,

then $\quad$ AOB is a straight line.

*Abbreviation for reference:* adj. $\angle$s. supp.

**Definition.** Three or more points are said to be **collinear** if they lie on the same straight line.

In fig. 206, if $a + b = 2$ rt. $\angle$s, then A, O, B are collinear.

## THEOREM 3

**If two straight lines intersect, the vertically opposite angles are equal.**

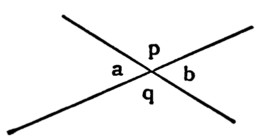



FIG. 207

With the notation of fig. 207, which represents two intersecting straight lines,

$$a = b \quad \text{and} \quad p = q.$$

*Abbreviation for reference:* vert. opp. $\angle$s.

*A formal proof was given on p.* 28.

### Examples for Oral Discussion

**1.** Draw a figure to show how fig. 205, p. 86, must be constructed (i) if $e + a + b = 2$ rt. $\angle$s;
(ii) if $b + c + d = 2$ rt. $\angle$s;
(iii) if $a + b + c = 2$ rt. $\angle$s.

Draw four lines AH, AK, AL, AM as in fig. 217, p. 90, *without* the numerical data and use it for Nos. 2–4:

**2.** If $\angle$KAL = $\angle$MAH, must LAH and KAM be straight lines?

**3.** If $\angle$KAL = $\angle$MAH and $\angle$HAK = $\angle$LAM, must LAH and KAM be straight lines?

**4.** If it is given that LAH is a straight line and that $\angle$KAL = $\angle$MAH, must KAM be a straight line?

## NUMERICAL EXAMPLES

### EXERCISE 24

**1.** Write down the supplements of 20°, 150°, 92°.

**2.** Find $x$ if $2x°$, $3x°$ are supplementary angles.

[3] Find $y$ if $4y° + 30°$, $y° + 40°$ are supplementary angles.

**4.** Find the size of an angle $z°$, if it is 4 times its supplement.

**5.** What is the least number of times you must turn through 17° in order to turn through (i) an obtuse angle, (ii) a reflex angle, (iii) more than one revolution?

[6] Through what angle does the minute-hand of a clock turn in (i) half an hour, (ii) 1 minute?

[7] Through what angle does the hour-hand of a clock turn in (i) 9 hours, (ii) 20 minutes?

***8.** Find the angle between the hands of a clock (i) at 7 o'clock, (ii) at 20 minutes past seven.

**9.** A wheel has 6 spokes equally spaced; what is the angle between 2 adjacent spokes?

[10] A wheel makes 40 revolutions per minute; through what angle does a spoke turn in 1 second?

Draw a figure representing the compass directions, north, south, east, west, etc. (see fig. 43, p. 23), and use it to find the angles between the following pairs of directions:—

11. N.E. and E.

12. E. and W.S.W.

13. S.W. and N.N.W.

[14] S.E. and W.

[15] S. and N.N.E.

[16] E.S.E. and W.S.W.

17. 10° E. of N. and 30° E. of S.

[18] 72° E. of N. and 65° W. of N.

19. 20° W. of S. and 80° E. of N.

[20] 70° W. of N. and 10° E. of S.

21. Find the reflex angle between S. 80° E., N. 50° W.

*22. Two wheels A, B are geared so that A makes 6 revolutions when B makes 1 revolution. Through how many degrees does B turn when A makes $\frac{3}{4}$ of a revolution?

[23] In fig. 208, ACB is a straight line, find $x$.

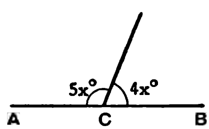



Fig. 208

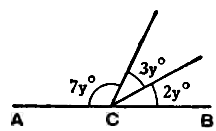



Fig. 209

24. In fig. 209, ACB is a straight line. Find $y$.

[25] In fig. 210, find $z$.

26. In fig. 211, find $x$ if $y = 40$.

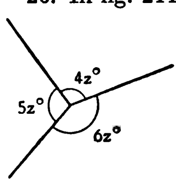



Fig. 210

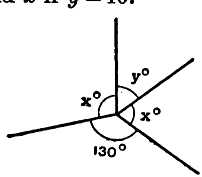



Fig. 211

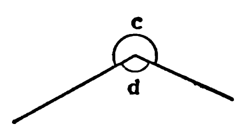



Fig. 212

[27] In fig. 211, find $y$ if $x = 90$.

28. In fig. 212, find $d$ if $c$ is twice $d$. [Suppose $d$ is $x°$.]

[29] In fig. 212, find $d$ if $c$ exceeds $d$ by 1 right angle

30. In fig. 213, ACB is a straight line. Find $a$ if $a$ is three times $b$. [Suppose $b$ is $x°$.]

[31] In fig. 213, ACB is a straight line. Find $a$ if $a$ exceeds $b$ by $\frac{1}{3}$ of a right angle.

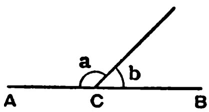



FIG. 213

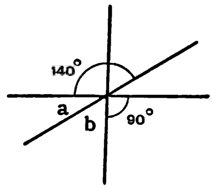



FIG. 214

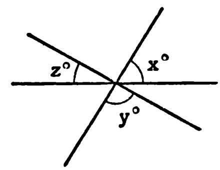



FIG. 215

32. Fig. 214 represents three straight lines intersecting at a point. Find $a$ and $b$.

[33] Fig. 215 represents three straight lines intersecting at a point. Find an equation connecting $x$, $y$, $z$.

[34] In fig. 216, not drawn accurately, (i) find ∠PAT; (ii) find the angle between the bisectors of ∠RAQ, ∠RAS.

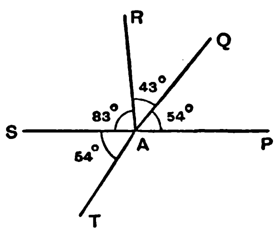



FIG. 216

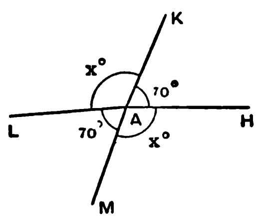



FIG. 217

35. In fig. 216, not drawn accurately, find three points which are collinear. Is there more than one answer? *Give reasons.*

36. In fig. 217, not drawn accurately,

(i) find the value of $x$;

(ii) find three points which are collinear. Is there more than one answer?

*Give reasons.*

**Use of Small Letters for Angles.** Proofs can often be written down more shortly by using small letters to represent angles.

It is also a help to use a notation for two angles which are given equal or can easily be proved to be equal which suggests this equality, e.g. $a$ and $a_1$, or $a_1$ and $a_2$, etc.

*Whenever you use small letters for angles, draw a* large *diagram.* Unless you do so, the notation will not be clear.

**Example for Oral Discussion**

**If a straight line EC meets another straight line ACB at C and if CH, CK are the bisectors of ∠ACE, ∠BCE, prove that ∠HCK is a right angle.**

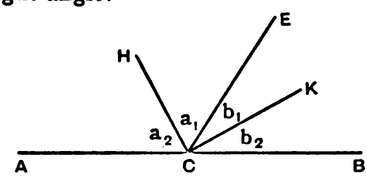



Fig. 218

Since ∠s HCE, HCA are given equal, we denote them by $a_1$, $a_2$; similarly for ∠s KCE, KCB.

(i) Express with small letters the fact that has to be proved.

(ii) What do you know about ∠ECA and ∠ECB? Give the reason. Express this fact with small letters.

(iii) What do you get by putting $a_2 = a_1$ and $b_2 = b_1$? Complete the proof.

**In proving a rider, always state all the necessary reasons.**

The proof of this rider may be set out as follows:—

$(a_1 + a_2) + (b_1 + b_2) = 2$ rt. ∠s *adj. ∠s on st. line,*

but $a_1 = a_2$ and $b_1 = b_2$ *given,*

$\therefore\ 2a_1 + 2b_1 = 2$ rt. ∠s,

$\therefore\ a_1 + b_1 = 1$ rt. ∠,

$\therefore$ ∠HCK $= 1$ rt. ∠.

## EXERCISE 25

Nos. 1–12 refer to fig. 219.

1. Express with small letters, (i) $\angle$POR, (ii) $\angle$QOS + $\angle$ROT.

[2] Express with small letters, (i) $\angle$QOT, (ii) $\angle$POS − $\angle$POR

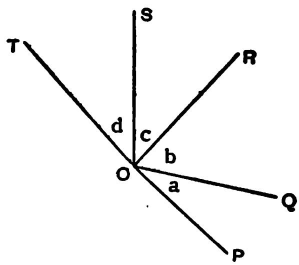



FIG. 219

3. Express with capital letters as simply as possible,
(i) $c+d$, (ii) $a+b+c$.

4. Express the following statements with small letters:
(i) OS bisects $\angle$ROT; (ii) TOP is a straight line.

[5] Express the following statements with small letters:
(i) $\angle$QOS = $\angle$ROT; (ii) OS is perpendicular to OQ.

6. If $\angle$POQ = $\angle$ROS, prove that $\angle$POR = $\angle$QOS.

[7] If $\angle$QOS = $\angle$ROT, prove that $\angle$QOR = $\angle$SOT.

[8] Prove that $\angle$POR + $\angle$QOS = $\angle$POS + $\angle$QOR.

9. If OR is perpendicular to OP, and if OS is perpendicular to OQ, prove that
(i) $\angle$POQ = $\angle$ROS, (ii) $\angle$POS + $\angle$QOR = 2 rt. $\angle$s.

10. If OS bisects $\angle$ROT, prove that
$\angle$QOS − $\angle$TOS = $\angle$QOR.

*11. If OR bisects $\angle$QOT, prove that
$\angle$ROS = $\frac{1}{2}$($\angle$QOS − $\angle$TOS).

*12. If $\angle$POT = 2$\angle$QOS and if OR bisects $\angle$POT, prove that
$\angle$POQ = $\angle$ROS.

Nos. 13–15 refer to fig. 220, in which OM, ON are the *bisectors* of $\angle$ EOH, $\angle$ EOK.

13. Prove that $\angle$ MON $=\frac{1}{2}\angle$ HOK.

[14] Prove that $\angle$ MOE + $\angle$ MOK = 2$\angle$ MON.

15. Prove that $\angle$ HON + $\angle$ KOM = 3$\angle$ MON.

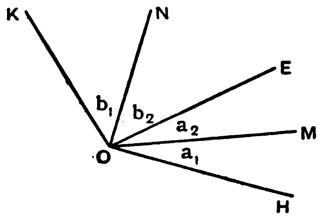



FIG. 220

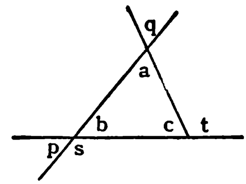



FIG. 221

Nos. 16–18 refer to fig. 221, which represents three intersecting straight lines. *State the reasons clearly.*

16. If $p=q$, prove that $a=b$.

[17] If $b=c$, prove that $s=t$.

18. If $a=c$, prove that $q+t=2$ rt. $\angle$s.

19. In fig. 222, HCK is a straight line and CK bisects $\angle$ ACB. Prove that $s=t$.

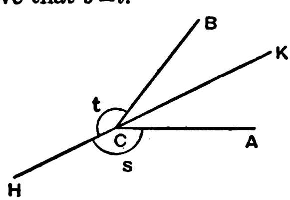



FIG. 222

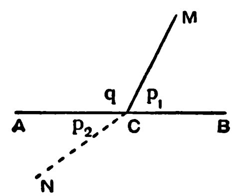



FIG. 223

20. In fig. 223, not drawn accurately, ACB is a straight line and $p_1=p_2$. Prove that MCN is a straight line.

*21. If in fig. 222 it is not *given* that HCK is a straight line, but it is given that $s=t$ and $\angle$ KCB = $\angle$ KCA, prove that HCK is a straight line.

*22. If in fig. 223 it is not *given* that ACB is a straight line, but it is given that $\angle$ BCM = $\angle$ ACN and that $\angle$ ACM = $\angle$ BCN, prove that ACB and MCN are straight lines.

### Parallels and Transversals

**Definition.** If two straight lines are **coplanar**, *i.e.* lie in the same plane, and if they never meet however far they are produced either way, they are called **parallel straight lines.**

The following assumption about parallel straight lines is called **Playfair's Axiom.**

*Through a given point, one and only one straight line can be drawn parallel to any given straight line which does not pass through the given point.*

**Definition.** If a straight line cuts two or more other straight lines, it is called a **transversal.**

The meanings of the words, *alternate*, *corresponding*, *interior*, applied to angles formed by a transversal with the lines it cuts, have been given on p. 35.

In fig. 224,

$a$ and $b$ are called **alternate** angles,
$c$ and $b$ are called **corresponding** angles,
$d$ and $b$ are called **interior** angles on the same side of the transversal, or shortly interior angles; they are also called **allied** angles.

The use of pairs of these angles in providing tests for lines to be parallel, and the relations between pairs of these angles formed by a transversal cutting two parallel lines have been discussed in detail in Stage A, see pp. 35–38. These results are expressed by the following theorems.

## THEOREM 5

**Two straight lines are parallel if a transversal makes**

**(i) a pair of alternate angles equal,**

**or (ii) a pair of corresponding angles equal,**

**or (iii) a pair of interior angles on the same side of the transversal supplementary.**

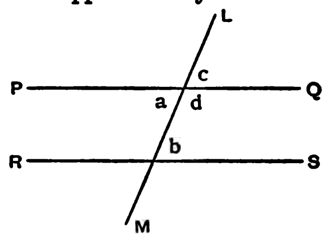



FIG. 224

With the notation of fig. 224, PQ || RS,

if (i) $a=b$,

or if (ii) $c=b$,

or if (iii) $b+d=2$ rt. $\angle$s.

*Abbreviations for reference:* (i) alt. $\angle$s equal,
(ii) corr. $\angle$s equal,
(iii) int. $\angle$s supp.

*A formal proof of Theorem* 5 (*i*) *is given in the Appendix, p.* 544.

Theorems 5 (ii), 5 (iii) can be deduced from Theorem 5 (i).

(1) If $c=b$, then $a=b$.

$a=c$ *vert. opp.* $\angle$*s,*
but $c=b$ *given.*
$\therefore a=b$

(2) If $b+d=2$ rt. $\angle$s, then $a=b$.

$a+d=2$ rt. $\angle$s *adj.* $\angle$*s on st. line,*
but $b+d=2$ rt. $\angle$s *given.*
$\therefore a+d=b+d$,
$\therefore a=b$.

## THEOREM 6

**If a transversal cuts two parallel straight lines,**

**(i) alternate angles are equal,**
**(ii) corresponding angles are equal,**
**(iii) interior angles on the same side of the transversal are supplementary.**

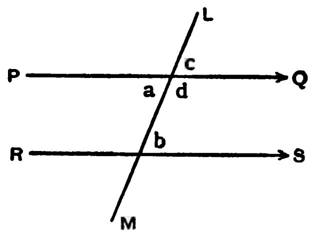



FIG. 225

With the notation of fig. 225, in which PQ ∥ RS,

(i) $a = b$,
and (ii) $c = b$,
and (iii) $b + d = 2$ rt. ∠s.

*Abbreviations for reference:* (i) alt. ∠s, PQ ∥ RS;
(ii) corr. ∠s, PQ ∥ RS;
(iii) int. ∠s, PQ ∥ RS.

*A formal proof is given in the Appendix, p.* 545.

NOTE. Markings in the diagrams for Theorems refer only to what is known from the data and construction, and not to any new facts discovered in the course of the proof.

## THEOREM 7

**Coplanar straight lines which are parallel to the same straight line are parallel to one another.**

This follows from Playfair's Axiom, because two intersecting straight lines cannot both be parallel to a third straight line.

Theorem 7 is also true if the straight lines are not all coplanar, as, for example, in the case of parallel edges of a cuboid. The general statement is as follows:

**If two straight lines are each parallel to a third straight line, the three lines not being coplanar, they are parallel to one another.**

## NUMERICAL EXAMPLES

### EXERCISE 26

[*Arrows indicate that lines are given parallel.*]

**State the reasons clearly in each example.**

Find the values of $x$ in figs. 226–228:

1.

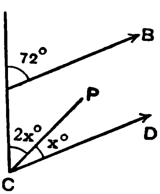



FIG. 226

2.

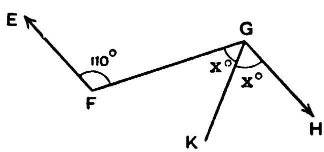



FIG. 227

[3]

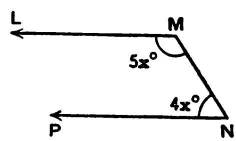



FIG. 228

Find the values of $x$, $y$ in figs. 229, 230:

4.

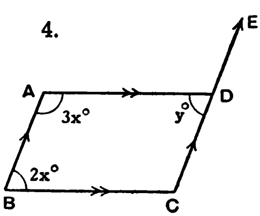



FIG. 229

*5.

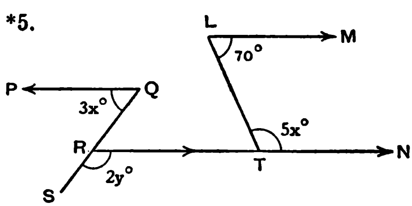



FIG. 230

Find the unknown marked angles in figs. 231–234. Draw your own figure and insert another parallel line in each case.

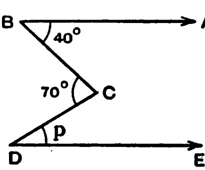
6. 


Fig. 231

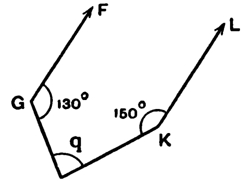
[7] 


Fig. 232

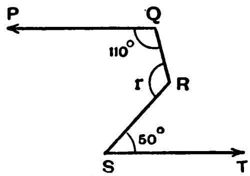
8. 


Fig. 233

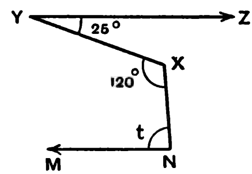
[9] 


Fig. 234

10. In fig. 235, prove that CD is parallel to BE.

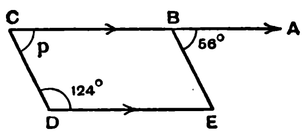



Fig. 235

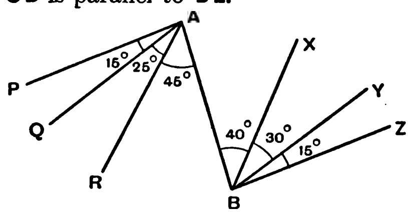



Fig. 236

[11] Find pairs of parallel lines in fig. 236.

*12. In fig. 237, ABCD is a straight line; prove BK || CP.

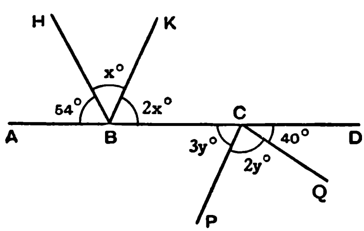



Fig. 237

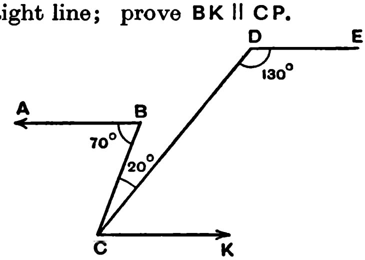



Fig. 238

*13. In fig. 238, prove that AB is parallel to DE.

## EXERCISE 27

*[Arrows indicate that lines are given parallel.]*

In the following examples draw your own figure and mark on it any necessary construction. **State the reasons clearly.**

1. In fig. 239, prove that $p = q$. [No construction.]

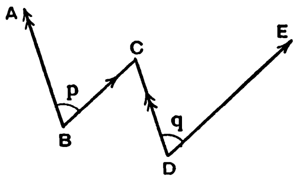


FIG. 239

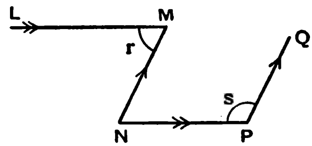


FIG. 240

2. In fig. 240, prove that $r + s = 2$ rt. $\angle$s. [No construction.]

[3] In fig. 241, prove that $m = n$. [Produce DE to meet BC at K.]

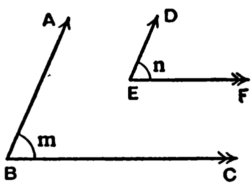


FIG. 241

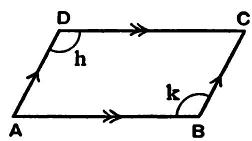


FIG. 242

4. In fig. 242, prove that $h = k$. [Produce BC to E.]

5. Draw any triangle ABC and produce BC to D. Prove that $\angle ACD = \angle A + \angle B$. [Draw CK parallel to BA.]

[6] Draw a triangle ABC in which $\angle B = \angle C$. Produce BA to E. Draw AP parallel to BC. Prove that AP bisects $\angle CAE$. [No construction.]

7. In fig. 243, CBK is a straight line and BQ bisects $\angle ABK$; prove that $\angle A = \angle C$. [No construction.]

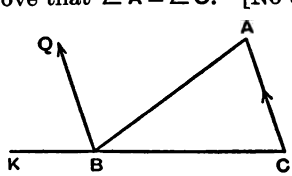


FIG. 243

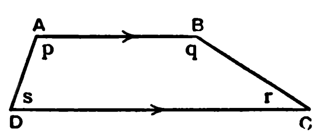


FIG. 244

8. In fig. 244, prove that $p - r = q - s$. [No construction.]

[9] In fig. 245, prove that $\angle BCD = p + q$. [Draw CN parallel to ED.]

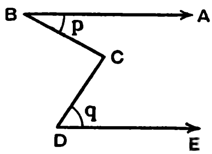


FIG. 245

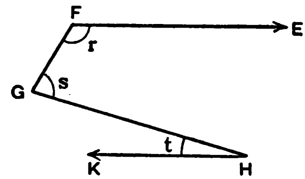


FIG. 246

*10. In fig. 246, prove that $r + s - t = 2$ rt. $\angle$s.

*11. If in fig. 245, when BD is joined, BC and DC are the bisectors of $\angle ABD$ and $\angle EDB$, use the fact proved in No. 9 to show that $\angle BCD$ is a right angle.

*12. Draw a figure like fig. 238, p. 98, but *omitting* the *numerical* measurements. If AB is parallel to DE, prove that

$$\angle ABC + \angle CDE - \angle BCD = 2 \text{ rt. } \angle\text{s.}$$

## Formal Proofs

In Stage A, various fundamental geometrical facts, called *theorems*, have been discussed and illustrated practically, but proofs have not been given. The proofs of some of these facts are given in an appendix to which reference may be made at a later stage. But for the present the reader should *assume* the truth of the fundamental facts given about—

(i) angles at a point,
(ii) angles made by a transversal with parallel lines,
(iii) angle-tests for lines to be parallel,
(iv) tests for triangles to be congruent.

By making use of these assumptions, it is then possible to give systematic proofs of other important theorems.

A general statement of the fact which it is required to prove is called the **general enunciation** of the theorem. If this fact is stated in terms of the letters of a particular diagram, it is called the **particular enunciation.**

## Writing out the Proof of a Theorem

If what is required to be proved is expressed in the form of a *general enunciation*, set out the work in the following order:

Using the letters of your figure,

(1) state what is given;
(2) state what it is required to prove;
(3) state the construction, if any is necessary;
(4) state the proof; this must include all necessary reasons, using suitable abbreviated references.

*Never give the number of a theorem as a reference.*

For example, the general enunciation of Theorem 8 is printed at the top of p. 102, and the argument which follows the diagram is expressed in terms of the letters of the diagram and contains the following stages:

(1) **Given.** (2) **To Prove.** (3) **Construction.** (4) **Proof.**

It is necessary to distinguish carefully between what is *given in the figure* and what is *added to the figure* for purposes of proof. Thus in Theorem 8, the fact that BC is produced to D is part of what is given and must be included in the particular enunciation because it is needed for stating what has to be proved; it is *not* part of the "construction," *i.e.* it is not an addition to the figure which is made to help in the proof.

If the particular enunciation is given, the letters used in the figure must be those in the given enunciation, but it is then unnecessary to repeat over again the statements of what is given and what has to be proved, because this is merely equivalent to copying the printed question.

**Examination Requirements.** In Theorem 8, the enunciation contains two separate statements, and the proof of the second depends on that of the first. If in an examination a candidate is asked to prove only the second statement, full marks are usually not given unless the proof of the first statement is also included. Throughout this book, the theorems are so arranged that this remark applies to all enunciations which contain more than one part.

## THEOREM 8

**(1) If one side of a triangle is produced, the exterior angle so formed is equal to the sum of the interior opposite angles.**

**(2) The sum of the angles of a triangle is equal to two right angles.**

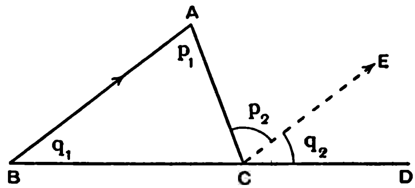



FIG. 247

**Given** a triangle ABC with BC produced to D.

**To prove that** (i) $\angle ACD = \angle A + \angle B$;
(ii) $\angle A + \angle B + \angle ACB = 2$ right angles.

**Construction.** Through C draw CE $\parallel$ BA.

**Proof.** (i) With the notation in the figure,

$p_2 = p_1$ *alt.* $\angle s$, CE $\parallel$ BA,
$q_2 = q_1$ *corr.* $\angle s$, CE $\parallel$ BA,

$\therefore$ by addition, $p_2 + q_2 = p_1 + q_1$,
$\therefore$ $\angle ACD = \angle A + \angle B$.

(ii) Add $\angle ACB$ to each side,
$\therefore$ $\angle ACD + \angle ACB = \angle A + \angle B + \angle ACB$.

But $\angle ACD + \angle ACB = 2$ rt. $\angle$s. *adj.* $\angle s$ *on st. line*,
$\therefore$ $\angle A + \angle B + \angle ACB = 2$ rt. $\angle$s.

**Corollary 1. If two triangles have two angles of the one equal to two angles of the other, each to each, then the third angles are also equal.**

**Corollary 2. In a right-angled triangle, (i) the right angle is the greatest angle, (ii) the sum of the two remaining angles is 1 right angle, and each of these angles is acute.**

**Corollary 3. At least two of the angles in every triangle are acute.**

**Corollary 4. From a given point outside a given line, not more than one straight line can be drawn perpendicular to the given line.**

*Abbreviations for reference:* For Theorem 8 (1), ext. $\angle$ of $\triangle$.
For Theorem 8 (2), $\angle$ sum of $\triangle$.

NOTE. The corollaries form suitable examples for oral discussion. Exercise 29, Nos. 4–8 illustrate the usefulness of Corollary 1.
Corollary 4 is used on p. 171.

### Examples for Oral Discussion

**1. ABC is a triangle in which $\angle A + \angle B = \angle C$, prove that $\angle C$ is a right angle.**

*First Method.* Produce BC to D. Explain why $\angle ACD = \angle ACB$, and complete the proof.

*Second Method.* What do you know about $\angle A + \angle B + \angle C$?

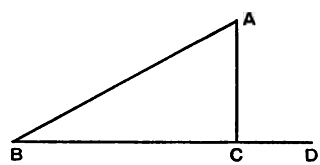


FIG. 248

**2. Prove that from a given point P outside a given straight line BC, not more than one straight line can be drawn perpendicular to BC.**

If possible, suppose that two different lines PM, PN can be drawn both perpendicular to BC. Explain why PMN and PNM cannot both be right angles.

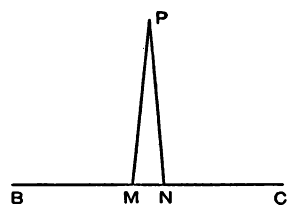


FIG. 249

**N.B.** The tendency to avoid using the exterior angle property of a triangle and to rely only on the angle-sum property *must be checked from the start.* It is often advisable to give extra credit to those pupils who use the exterior angle property when it is shorter to do so.

## NUMERICAL EXAMPLES

### EXERCISE 28

[*Use the property of the exterior angle instead of the property of the angle-sum of a triangle whenever it is shorter to do so.*]

Find the value of $x$ and the angle $a$ in figs. 250–252:

1. [2] 3.

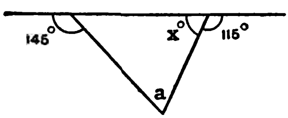



FIG. 250

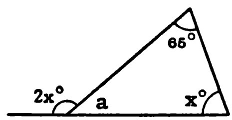



FIG. 251

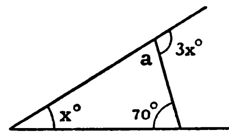



FIG. 252

4. In a right-angled triangle, one angle is 43°; find the other acute angle.

Nos. 5–12 refer to the angles A, B, C of a triangle ABC:

5. Find C if A = 40°, B = 105°; also if A = 90°, B = $x$°.

[6] Find B if A = 15°, C = 18°.

[7] Find A if B = C = 52°.

8. Find A if A = B and C = 48°.

9. Find C if A + B = 3C.

*10. Find A if A − B = 15° and B − C = 30°.

*11. Express $\frac{1}{2}$A + $\frac{1}{2}$B in terms of C.

*12. Simplify $\frac{1}{2}$(B + C − A).

13. Can a triangle be drawn having its angles equal to (i) 45°, 65°, 80°; (ii) 43°, 64°, 73°; (iii) 100°, 110°, $x$°?

[14] The angles of a triangle are $2x$°, $3x$°, $4x$°. Find $x$.

15. The angles of a triangle are $y$°, $2y° - 20°$, $3y° - 40°$. Find $y$.

16. Two exterior angles of a triangle are 120°, 130°. Find the third unequal exterior angle.

Find the unknown marked angles in figs. 253–257. Arrows indicate that lines are given parallel.

17.

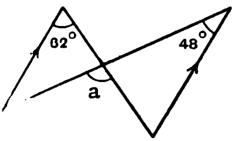



Fig. 253

[18]

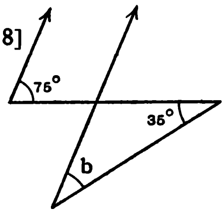



Fig. 254

19.

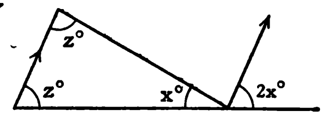



Fig. 255

[20]

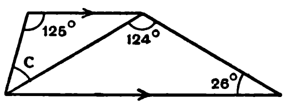



Fig. 256

21.

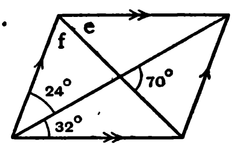



Fig. 257

22. In △ABC, AX bisects ∠BAC and AD is perpendicular to BC. If ∠A = 60° and ∠B = 70°, find ∠DAX.

[23] In △ABC, ∠B = 110°, ∠C = 50°; AD is the perpendicular from A to CB produced. Prove that ∠DAC = 2∠DAB.

24. In △ABC, ∠B = 90°, ∠A = 2∠C, and the bisector of ∠A cuts BC at K. Prove that ∠AKC = 4∠C.

Nos. 25–29 refer to fig. 258, in which BH, CK are the bisectors of ∠ABC, ∠ACB.

[25] If ∠A = 80°, ∠ABC = 30°, find ∠BIC.

26. If ∠ABC = 44°, ∠BIC = 125°, find ∠BHC.

*27. If ∠A = 74°, ∠ACB = 66°, and if the perpendicular AD from A to BC cuts BH, CK at P, Q, find the angles of △IPQ.

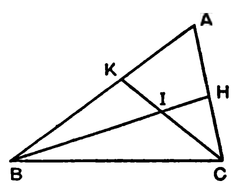



Fig. 258

*28. If ∠BKC = 95°, ∠BHC = 82°, find ∠A.

*29. If ∠BIC = 132°, find ∠A.

## EXERCISE 29

1. If, in fig. 259, $p = 2c$, prove that $b = c$.

[2] If, in fig. 259, $b = 2c$, prove that $c = \frac{1}{3}p$.

3. In fig. 260, BA is parallel to CD, prove that $r = d + e$.

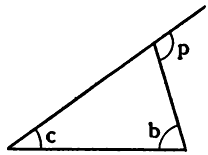


FIG. 259

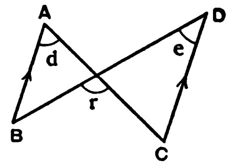


FIG. 260

4. ABCD is a quadrilateral such that the diagonal AC bisects ∠DAB and bisects ∠DCB. Prove that ∠ABC = ∠ADC.

[5] In fig. 261, BE and CF are perpendicular to AC and AB. Prove that ∠FBE = ∠FCE.

6. In fig. 261, BE and CF are perpendicular to AC and AB. Prove that (i) ∠FHB = ∠BAC; (ii) ∠BHC, ∠BAC are supplementary.

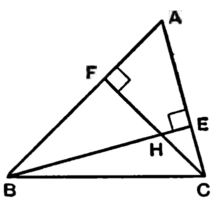


FIG. 261

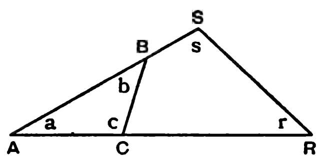


FIG. 262

7. In fig. 262, if $b = r$, prove that $c = s$.

[8] In fig. 262, find $r$ in terms of $b$, $c$, $s$.

9. In fig. 263, if $n_1 = n_2$, prove that $q = r$.

[10] In fig. 263, if $n_1 = n_2$, prove that ∠ACD = ∠BAD. [Put in a small letter for ∠BAC.]

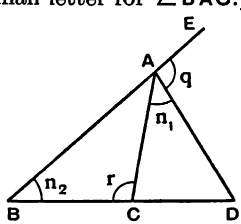


FIG. 263

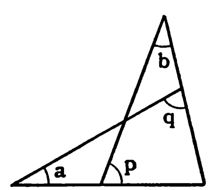


FIG. 264

[11] In fig. 264, if $a = b$, prove that $p = q$.

12. In fig. 264, find $q$ in terms of $a$, $b$, $p$.

13. In fig. 265, find $t$ in terms of $a$, $b$, $c$.

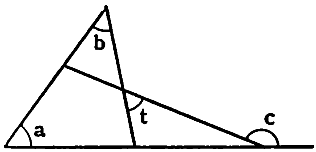



FIG. 265

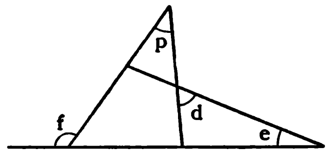



FIG. 266

[14] In fig. 266, find $p$ in terms of $d$, $e$, $f$.

15. Draw a triangle ABC right-angled at A and draw the perpendicular AD from A to BC. What angle in your figure is equal to $\angle$DAC? Give reasons.

16. In fig. 267, prove that $p+q+r=4$ right angles.

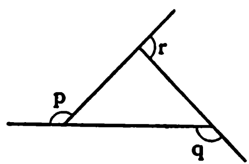



FIG. 267

[17] If, in fig. 267, $p+q=3r$, prove that the triangle is right-angled.

18. ABCD is a parallelogram; the bisectors of $\angle$BAD, $\angle$ABC meet at P; prove that $\angle$APB is a right angle.

*19. If, in the triangle ABC, the bisectors of the angles ABC, ACB meet at I, prove that $\angle BIC = 90° + \frac{1}{2}\angle BAC$.

*20. The side BC of $\triangle$ABC is produced to D; the bisector of $\angle$BAC cuts BC at K; prove that $\angle ABD + \angle ACD = 2\angle AKD$.

*21. The side BC of $\triangle$ABC is produced to D; the bisectors of the angles ABC, ACD meet at Q; prove that $\angle BQC = \frac{1}{2}\angle BAC$.

*22. ABCD is a tetrahedron (*i.e.* a pyramid on a triangular base). If the plane angles at each of the corners A, B, C add up to two right angles, prove that the plane angles at the corner D add up to two right angles. Sketch the net of ABCD.

## Sum of the Angles of a Polygon

If we wish to find the sum of the angles of a quadrilateral ABCD, the simplest method is to draw a diagonal AC, thus forming the two triangles ABC, ACD.

With the notation in fig. 268,

from $\triangle$ABC, $m+n+p=2$ rt. $\angle$s, $\angle$ *sum of* $\triangle$,

from $\triangle$ACD, $r+s+t=2$ rt. $\angle$s, $\angle$ *sum of* $\triangle$;

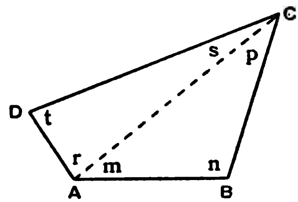


FIG. 268

but the angles of ABCD are $r+m$, $n$, $p+s$, $t$.

$\therefore$ by addition, the sum of the angles of the quadrilateral ABCD is 4 right angles.

Similarly, if we wish to find the sum of the angles of a pentagon ABCDE, the simplest method is to draw the two diagonals AC, AD, thus forming the three triangles ABC, ACD, ADE.

Hence it follows that the sum of the angles of a pentagon is 6 right angles.

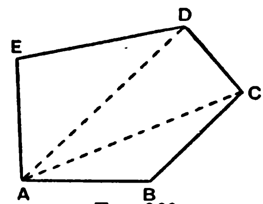


FIG. 269

If, however, the polygon has a large number of sides it is better to use a different method.

## Examples for Oral Discussion

1. Find the sum of the interior angles of the hexagon ABCDEF.

Take any point O inside the hexagon; join it to each vertex.

Copy and complete the following:

The total sum of all the angles of the 6 triangles OAB, OBC, OCD, ODE, OEF, OFA, is . . .; but the sum of the 6 angles at O is . . .;

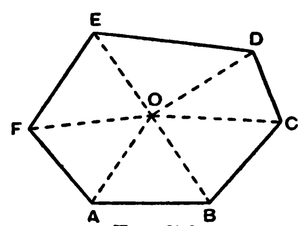


FIG. 270

$\therefore$ the sum of the interior angles of ABCDEF is . . .

2. Repeat No. 1 for a 7-sided polygon.

3. Work out the sum of the interior angles of (*a*) a 10-sided polygon, (*b*) a 100-sided polygon, (*c*) an *n*-sided polygon.

4. If the sides of a convex pentagon ABCDE are produced in order, *i.e.* as shown in fig. 271, find the sum of the exterior angles so formed.

Copy and complete the following:

The interior $\angle$ + the exterior $\angle$ at each vertex = 2 rt. $\angle$s;

$\therefore$ the sum of the 5 int. $\angle$s + the sum of the 5 ext. $\angle$s = . . . ;

but the sum of the 5 int. $\angle$s = . . . ;

$\therefore$ the sum of the 5 ext. $\angle$s = . . .

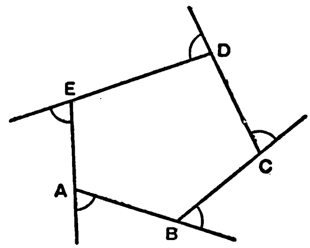


FIG. 271

5. Repeat No. 4 for a convex 7-sided polygon.

6. Work out the sum of the 100 exterior angles of a convex 100-sided polygon. Repeat for an *n*-sided polygon.

7. Draw a large convex pentagon ABCDE on the ground. Start at any point K on AB and walk along KB to B, then turn and walk along BC to C, and so on until you have arrived back at K. What are the separate angles through which you have turned? What is the total angle?

8. Apply the argument of No. 7 to a pentagon ABCDE which is not convex, see fig. 272.

Copy and complete the following:

The angles turned through are $p$, $q$, $r$, $t$ counter-clockwise, and $s$ clockwise.

But the total angle turned through is . . .

$\therefore p + q + \ . \ . \ . = \ . \ . \ .$

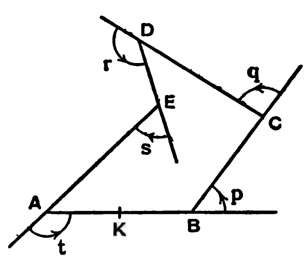


FIG. 272

These examples illustrate the fact that the sum of the exterior angles of a polygon formed by producing the sides in order is 4 right angles *if the polygon is convex*, but not otherwise.

# NUMERICAL EXAMPLES

## EXERCISE 30

1. Three of the angles of a quadrilateral are 80°, 100°, 110°; find the other angle.

[2] Three of the angles of a quadrilateral are 112°, 75°, 51°; find the other angle.

[3] ABCD is a quadrilateral in which $\angle A = \angle B = \angle C$, and $\angle D = 120°$; find $\angle A$.

4. The angles of a quadrilateral taken in order are $2x°$, $3x°$, $7x°$, $8x°$. Find the value of $x$ and prove that two of the opposite sides are parallel.

5. The angles of a pentagon are $x°$, $2x°$, $2x°$, $2x°$, $3x°$; find the value of $x$.

[6] The angles of a pentagon are $x$, $2x$, $x+30$, $x-10$, $x+40$ degrees; find the value of $x$.

7. ABCDE is a regular pentagon; AB, DC are produced to meet at P; find $\angle BPC$.

8. Find the sum of the interior angles of a polygon which has (i) 30 sides, (ii) 40 sides.

[9] Prove that the sum of the interior angles of an octagon (8 sides) is twice the sum of the interior angles of a pentagon.

10. Find the size of an exterior angle of (i) a regular octagon (8 sides), (ii) a regular decagon (10 sides).

[11] Find the size of an exterior angle of a regular polygon having (i) 15 sides, (ii) $n$ sides.

12. Find the number of sides of a polygon (i) if each exterior angle is 40°, (ii) if each interior angle is 144°.

[13] Find the number of sides of a polygon (i) if each exterior angle is 15°, (ii) if each interior angle is 160°.

14. If the sum of the interior angles of a polygon is 30 right angles, find the number of its sides.

[15] Five of the angles of a hexagon are equal to each other and each is greater than the remaining angle by 72°. Find the size of the smallest angle.

16. Find the sum of the interior angles of the re-entrant pentagon APBCD in fig. 273.

[17] If seven of the angles of an octagon are each 150°, find the remaining angle and prove that there are two pairs of parallel sides.

18. In fig. 273, PA, PB, RC, RD are the bisectors of the angles of the quadrilateral ABCD. If $\angle DAB = 80°$, $\angle ABC = 70°$, $\angle BCD = 150°$, prove that the opposite angles of the quadrilateral PQRS are supplementary.

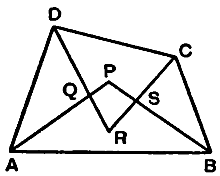



FIG. 273

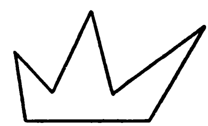


FIG. 274

[19] The sides of a quadrilateral ABCD are produced in order and the bisectors of the exterior angles at each vertex are drawn to form a quadrilateral LMNP. If $\angle DAB = 64°$, $\angle ABC = 80°$, $\angle BCD = 142°$, find the angles of LMNP. What do you notice about them?

20. Find the sum of the interior angles of the re-entrant polygon in fig. 274.

21. Find the number of sides of a polygon if the sum of its angles is three times that of an octagon.

*22. If the angles of a quadrilateral, taken in order, are in the ratios $1:3:5:7$, prove that two of its sides are parallel.

*23. If the angles of a pentagon, taken in order, are in the ratios $4:8:6:4:5$, prove that two pairs of sides are parallel.

*24. ABCDE is a pentagon in which AE is parallel to BC; $\angle C - \angle A = 40°$; $\angle D - \angle B = 30°$. Find (i) $\angle C + \angle D$; (ii) $\angle E$.

*25. If each interior angle of a regular polygon is $p$ times as large as each exterior angle, prove that the polygon has $2(p+1)$ sides.

*26. A solid is bounded by F plane faces and has E edges. Prove that the sum of all the angles of all the faces is $4(E-F)$ right angles.

## THEOREM 9

**(1) The sum of the interior angles of a convex polygon of $n$ sides is $(2n-4)$ right angles.**

**(2) If the sides of a convex polygon are produced in order, the sum of the exterior angles so formed is four right angles.**

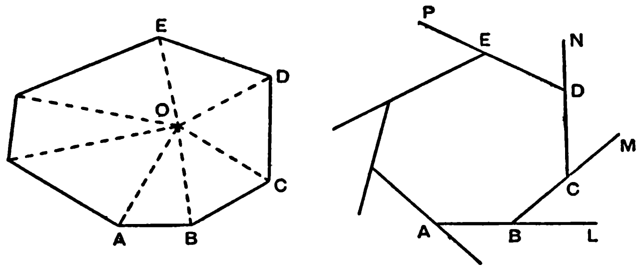



FIG. 275

**Given** a convex polygon ABCDE . . . of $n$ sides AB, BC, CD, DE, . . . produced to L, M, N, P, . . .

**To prove that** (1) the sum of the interior angles ABC, BCD, . . . is $(2n-4)$ right angles.
(2) the sum of the exterior angles LBC, MCD, . . . is 4 right angles.

**Construction.** Take any point O inside the polygon and join it to each vertex.

**Proof.** (1) The polygon has $n$ sides and therefore has been divided into $n$ triangles by the construction.

The sum of the angles of each triangle is 2 rt. $\angle$s,

$\therefore$ the sum of the angles of the $n$ triangles is $2n$ rt. $\angle$s,

$\therefore$ sum of $\angle$s of polygon + sum of $\angle$s at O $= 2n$ rt. $\angle$s;
but sum of $\angle$s at O $= 4$ rt. $\angle$s,

$\therefore$ sum of $\angle$s of polygon $+ 4$ rt. $\angle$s $= 2n$ rt. $\angle$s.

$\therefore$ sum of $\angle$s of polygon $= (2n-4)$ rt. $\angle$s.

(2) At each vertex,

interior $\angle$ + exterior $\angle = 2$ rt. $\angle$s. *adj. $\angle$s on st. line,*
and there are $n$ vertices.

Therefore, if we denote the sum of all the interior $\angle$s by X, and the sum of all the exterior $\angle$s by Y,

$$X + Y = 2n \text{ rt. } \angle\text{s};$$

but $\quad X + 4 \text{ rt. } \angle\text{s} = 2n \text{ rt. } \angle\text{s}$ *proved,*

$$\therefore\ X + Y = X + 4 \text{ rt. } \angle\text{s},$$

$$\therefore\ Y = 4 \text{ rt. } \angle\text{s}.$$

*Abbreviations for reference:* (1) $\angle$ sum of polygon.
(2) Sum of ext. $\angle$s of polygon.

N.B. If a step in the proof of a theorem depends on a *previous* theorem, the reason must be given in words or symbols. To say "proved" as a reason, as is done in Theorem 9, is only allowed if it refers to a previous part of the proof itself.

Theorem 9 (1) is also true for a re-entrant polygon, but it may be necessary to modify the proof. For example, it is impossible to find a point O inside the 8-sided re-entrant polygon in fig. 276 such that, when O is joined to each vertex, the polygon is divided into 8 triangles.

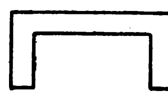


FIG. 276

## For Oral Work

(1) What is the sum of the interior angles of the polygon in fig. 276 if each corner is right-angled?

(2) How could you find the sum of the interior angles of the polygon in fig. 276 if the corners are not right-angled?

Theorem 9 (2) is true only if the polygon is convex.

## EXERCISE 31

**1.** ABCD is a quadrilateral in which $\angle A = \angle C$ and $\angle B = \angle D$. Prove that (i) $\angle A + \angle B = 2$ rt. $\angle$s; (ii) ABCD is a parallelogram.

[2] ABCD is a quadrilateral in which $\angle A + \angle C = 2$ rt. $\angle$s If AB is produced to E, prove that $\angle CBE = \angle D$.

**3.** In fig 277, NA, NB, NC, ND are the bisectors of the angles of the quadrilateral ABCD. Prove $\angle ANB + \angle CND = 2$ rt. $\angle$s.

[4] If in fig. 277, NA and NB are the bisectors of $\angle DAB$, $\angle CBA$, prove that $\angle ADC + \angle BCD = 2\angle ANB$.

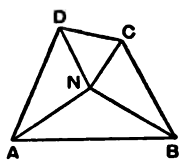



FIG. 277

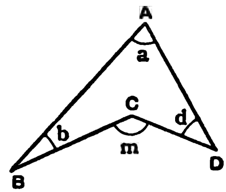



FIG. 278

**5.** ABCD is a quadrilateral in which $\angle A$ and $\angle C$ are each right angles. Prove that the bisectors of $\angle B$ and $\angle D$ are parallel.

**6.** In fig. 278, prove that $m = a + b + d$.

[7] If, in fig. 278, $m = 2a$ and $a = 2b$, prove that $b = d$.

**8.** If, in fig, 278, the bisectors of $\angle ABC$, $\angle ADC$ meet at K, prove that $a + m = 2\angle BKD$.

**9.** Find the sum of the marked angles in fig. 279.

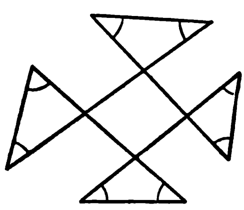


FIG. 279

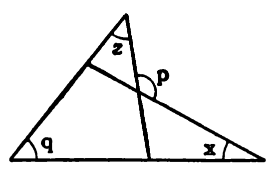



FIG. 280

***10.** If, in fig. 280, $p$ and $q$ are supplementary angles, prove that

$$p = 90° + \tfrac{1}{2}(x + z).$$

***11.** The sides of any convex hexagon are produced and the bisectors of the exterior angles so formed are drawn and produced to form another convex hexagon LMNPQR. Prove that $\angle L + \angle N + \angle Q = 4$ rt. $\angle$s.

## Tests for Congruence

Three general tests and one special test for the congruence of two triangles have been discussed in Stage A. The first of these, repeated here, is numbered Theorem 4, although printed after Theorem 9, because if at a later stage it is desired to reduce the number of assumptions made, this test for congruence is used to establish the test for parallel lines given in Theorem 5, p. 95. The method is given in the Appendix.

## THEOREM 4

**If two triangles have TWO SIDES of the one equal to TWO SIDES of the other, each to each, and also the angles INCLUDED by those sides equal, the triangles are congruent.**

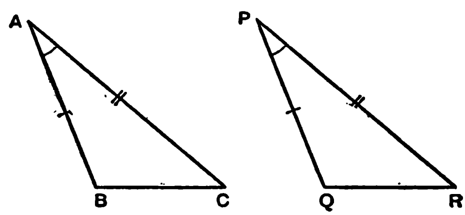



FIG. 281

In $\triangle$s ABC, PQR,

if $AB = PQ$, $AC = PR$ and $\angle A = \angle P$,

then $\triangle$s $\begin{matrix} ABC \\ PQR \end{matrix}$ are congruent.

*Abbreviation for reference:* SAS *or* 2 sides, inc. $\angle$.

*A formal proof of Theorem* 4 *is in the Appendix, p.* 542.

## THEOREM 10

**If two triangles have TWO ANGLES of the one equal to TWO ANGLES of the other, each to each, and also A SIDE of one equal to the CORRESPONDING SIDE of the other, the triangles are congruent.**

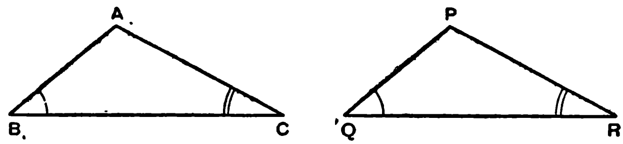


FIG. 282

In $\triangle$s ABC, PQR,

if $\angle B = \angle Q$ and $\angle C = \angle R$

and if *either* (i) $BC = QR$ *or* (ii) $AB = PQ$ *or* (iii) $AC = PR$,

then $\triangle$s $\begin{matrix} ABC \\ PQR \end{matrix}$ are congruent.

*Abbreviation for reference:* ASA, AAS *or* 2$\angle$s, corr. side.

*A formal proof of Theorem* 10 *is given in the Appendix, p.* 546.

NOTE. Since the sum of the angles of a triangle is two right angles, if two angles of one triangle are equal to two angles of another triangle, then the third angles are also equal.

The two sides which are given equal must be corresponding sides, *i.e.* they must be *opposite* to angles which are known to be equal.

## EXERCISE 32

[*Nos.* 1–8 *are suitable for oral discussion.*]

The data in Nos. 1–7 refer to two triangles ABC, DEF. In each case show the data *on your own figure* by suitable marking. Do the data show that the triangles *must* be congruent? If so, state the test used and express the fact in the proper way.

1. $AB = EF$, $AC = DF$, $\angle A = \angle F$.
2. $CA = FD$, $CB = FE$, $\angle B = \angle E$.
3. $AC = EF$, $\angle A = \angle E$, $\angle C = \angle F$.
4. $AC = DF$, $\angle B = \angle D$, $\angle C = \angle F$.
5. $BC = DF$, $\angle B = \angle F$, $\angle A = \angle E$.
6. $\angle A = \angle D$, $\angle B = \angle E$, $\angle C = \angle F$.
7. $AB = AC$, $DE = DF$, $\angle A = \angle D$.

8. If $\Delta$s $\begin{matrix}ABC\\YKX\end{matrix}$ are congruent, which angle is equal to $\angle C$? Which side is equal to XY?

9. If, in fig. 283, PR bisects $\angle QPS$ and $PS = PQ$, prove that $RS = RQ$.

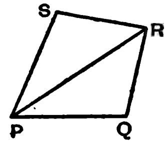


FIG. 283

10. If, in fig. 284, AB and CD bisect each other, prove that

(i) $AD = BC$, (ii) AD is parallel to CB.

11. If, in fig. 284, CB is parallel to AD and if N is the mid-point of AB, prove that $CN = ND$.

[12] If, in fig. 284, AD is equal and parallel to CB, prove that (i) N is the mid-point of AB, (ii) $AC = DB$, (iii) AC is parallel to DB.

[13] ABCD is a quadrilateral in which $AB = CD$ and $\angle B = \angle C$, prove that $AC = BD$.

14. A line AP is drawn bisecting the angle BAC; PX, PY are the perpendiculars from P to AB, AC. Prove that $PX = PY$.

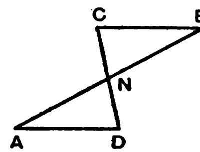


FIG. 284

[15] If the straight line XOY bisects at right angles the straight line AOB, prove that $XA = XB$.

16. ABCD is a quadrilateral in which AB is parallel to DC, and AD is parallel to BC. Prove that $AB = DC$ and that $AD = BC$. [Join AC.]

[17] O is the centre of each of the circles in fig. 285; AOB, POQ are straight lines. Prove that $AQ = PB$.

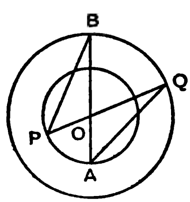


FIG. 285

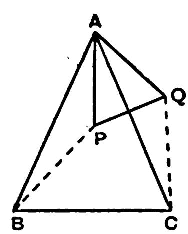


FIG. 286

18. In fig. 286, $AB = AC$, $AP = AQ$ and $\angle BAC = \angle PAQ$. Which two triangles in the figure are congruent? Give reasons.

[19] D is the mid-point of the base BC of $\Delta ABC$; BX, CY are perpendiculars from B, C to AD, produced if necessary. Prove that $BX = CY$.

20. X is the centre of each semicircle AQB, CPD in fig. 287; XPQ, ACXDB are straight lines.

(i) Which triangle in the figure is congruent to △AXP? Give reasons.

(ii) Name two other congruent triangles. Which angle is equal to ∠XPB? Give reasons.

(iii) Prove that

∠APB + CQD = 2 rt. ∠s.

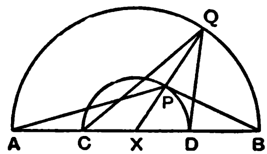


FIG. 287

21. In fig. 288, AB = AC and APQR is a straight line. If ∠BAC = $a_1 = a_2$, prove that AP = CQ.

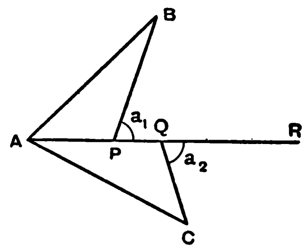


FIG. 288

*22. The perpendicular bisectors of the sides AB, AC of △ABC meet at O. Prove that OB = OC. [Join OA and prove that OB = OA.]

*23. The bisectors of ∠B, ∠C of △ABC meet at I; IQ, IR are the perpendiculars from I to AB, AC. Prove that IQ = IR. [Draw the perpendicular IP from I to BC.]

*24. AOB is a diameter of a circle, centre O: OP, OQ are two perpendicular radii; PH, QK are the perpendiculars from P, Q to AB. Prove that PH = OK.

Draw two figures, one in which P and Q are on the same side of AB and the other in which P and Q are on opposite sides of AB. Prove the fact in each case.

## Isosceles Triangles

If two sides of a triangle are equal, the triangle is called **isosceles**; the point of intersection of the equal sides is called the **vertex**, the angle included by the equal sides is called the **vertical angle** and the third side is called the **base** of the isosceles triangle.

If the three sides of a triangle are equal, the triangle is called **equilateral.**

If a triangle has three unequal sides, it is called **scalene.**

## THEOREM 11

**If two sides of a triangle are equal, then the angles opposite to those sides are equal.**

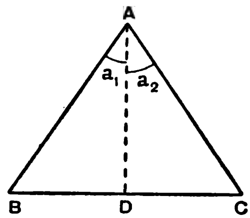



FIG. 289

**Given** a triangle ABC in which AB = AC.

**To prove that** $\angle$B = $\angle$C.

**Construction.** Let AD be the bisector of $\angle$BAC and let it meet BC at D.

**Proof.** In the $\triangle$s ABD, ACD,

AB = AC *given*,
AD = AD
$a_1 = a_2$ *constr.*,

$\therefore$ $\triangle$s $\begin{matrix}\text{ABD}\\\text{ACD}\end{matrix}$ are congruent SAS.

$\therefore$ $\angle$B = $\angle$C.

*Abbreviation for reference:* base $\angle$s, isos. $\triangle$.

Theorem 11 is often stated in the form:

**The angles at the base of an isosceles triangle are equal.**

## THEOREM 12

**If two angles of a triangle are equal, then the sides opposite to those angles are equal.**

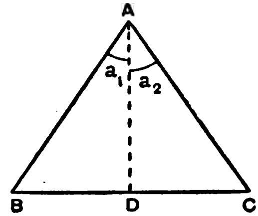



FIG. 290

**Given** a triangle ABC in which $\angle B = \angle C$.

**To prove that** $AB = AC$.

**Construction.** Let AD be the **bisector** of $\angle BAC$ and let it meet BC at D.

**Proof.** In the $\triangle$s ABD, ACD,

$\angle B = \angle C$ *given,*

$a_1 = a_2$ *constr.,*

$AD = AD$.

$\therefore$ $\triangle$s $\begin{matrix} ABD \\ ACD \end{matrix}$ are congruent AAS.

$\therefore$ $AB = AC$.

*Abbreviation for reference:* sides opp. equal $\angle$s.

**Important Note.** Theorem 11 is used to prove the SSS test for congruence (Theorem 13, p. 128) and the RHS test for congruence (Theorem 14, p. 130), and therefore neither of these tests can be used in proving Theorem 11. The construction used for the proof of Theorem 11 must therefore be noted carefully, because any different way of drawing AD would make the proof worthless. To avoid confusion, it is best to use the same construction for Theorem 12 as for Theorem 11.

The following deductions from Theorems 11, 12 are suggested as examples for oral discussion because they are often of use in rider-work.

### Examples for Oral Discussion

**1. If ABC is an equilateral triangle, prove that**

$$\angle A = \angle B = \angle C = 60°.$$

(i) Explain why $\angle A = \angle B$ and $\angle A = \angle C$.
(ii) What is the sum of $\angle A$, $\angle B$, $\angle C$?

**2. If ABC is a triangle in which $AB = AC$ and $\angle A = 60°$, prove that $\triangle ABC$ is equilateral.**

(i) What is the value of $\angle B + \angle C$?
(ii) What follows from the fact that $AB = AC$?

**3. If ABC is a triangle in which $\angle C = 90°$ and $\angle B = 60°$, prove that**

$$BC = \tfrac{1}{2} BA.$$

Produce BC to D making $BC = CD$.
Join AD.

(i) Explain why $\angle ACD$ is a right angle.
(ii) Prove that $\triangle ACB \equiv \triangle ACD$.
(iii) What is the size of $\angle D$?
(iv) Explain why $\triangle ABD$ is equilateral.

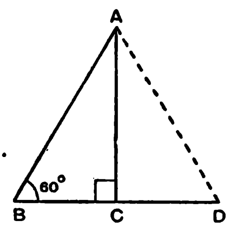


FIG. 291

NOTE. This result is best recognised and remembered by thinking of $\triangle ACB$ as "half an equilateral triangle."
It is equivalent to the statement,

$$\cos 60° = \tfrac{1}{2}$$

because $\cos ABC = \dfrac{BC}{BA}$. Since $\angle BAC = 30°$, $\sin 30° = \dfrac{BC}{BA} = \tfrac{1}{2}$.

**Converse Theorems.** If, in Theorem 11, the fact that is given and the fact to be proved are interchanged, the enunciation of Theorem 12 is obtained; for this reason Theorem 12 is called the **converse** of Theorem 11. Similarly, Theorem 11 is the converse of Theorem 12.

The fact that a theorem is true does not imply that the converse theorem is also true, although it often may be so. For example, the statement,

"If two angles are right angles, then they must be equal," *is true.*

But the converse statement,

"If two angles are equal, then they must be right angles," *is* **not** *true.*

## NUMERICAL EXAMPLES

### EXERCISE 33

[*Arrows indicate that lines are given parallel.*]

**1.** The vertical angle of an isosceles triangle is 110°; find the base angles.

[2] One base angle of an isosceles triangle is 62°; find the vertical angle.

**3.** One of the angles of an isosceles triangle is $x°$; find the other angles in terms of $x$. [Two sets of answers.]

**4.** Find the angles of an isosceles triangle if a base angle is double the vertical angle.

[5] Find the angles of an isosceles triangle if the vertical angle is three times a base angle.

**6.** In fig. 292, $AB = AC$ and $CX = CB$, find $\angle ACX$.

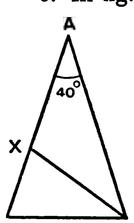


FIG. 292

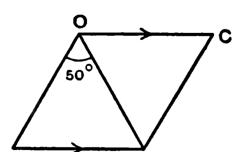


FIG. 293

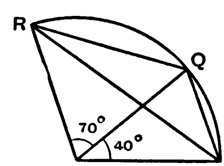


FIG. 294

**7.** In fig. 293, $OA = OB = OC$, find $\angle OCB$.

**8.** In fig. 294, O is the centre of the circular arc PQR. Find the angles of $\triangle PQR$.

[9] In fig. 295, AN = AC, find ∠ NAB.

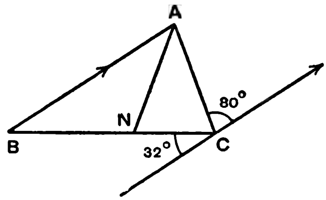



Fig. 295

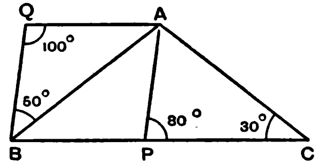



Fig. 296

10. In fig. 296, AB = AC and BPC is a straight line; prove that BP ∥ QA and BQ ∥ PA.

[11] In △ ABC, the bisector of ∠ A cuts BC at D. If AD = DB and if ∠ C = 66°, find ∠ B.

12. In fig. 297, ABCD is a straight line and BE = BC, prove that CE = CD.

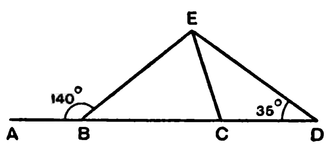



Fig. 297

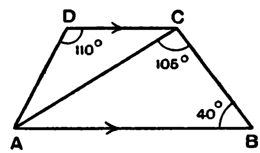



Fig. 298

13. In fig. 298, prove that DA = DC.

[14] The side BC of △ ABC is produced to D; ∠ BAC = 40°, ∠ ACD = 75°. P is a point on AB such that AP = AC, prove PB = PC.

15. In △ ABC, AB = AC; D is a point on AC produced such that BD = BA. If ∠ CBD = 36°, prove that BC = CD.

*In Nos.* 16–20, *work throughout in terms of the small letter which is to appear in the answer.*

16. In fig. 299, if AB = AC, find $x$ in terms of $y$.

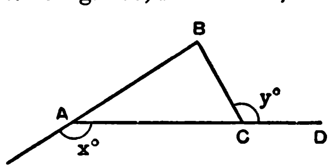



Fig. 299

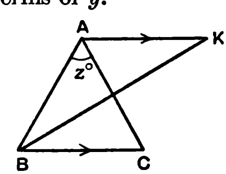



Fig. 300

17. In fig. 300, AB = AC and BK bisects ∠ ABC. Find ∠ AKB in terms of $z$.

*18. In fig. 301, AB = AC = BD. Find $y$ in terms of $x$.

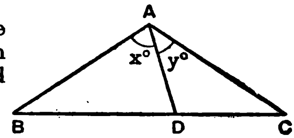


FIG. 301

*19. In $\triangle$ ABC, AB = AC; P, Q are points on AB, AC respectively such that AP = PQ. If $\angle$ ABC $= x°$, find $\angle$ BPQ in terms of $x$.

*20. ABCDE is a regular pentagon. Prove that the bisector of $\angle$BAC is perpendicular to AE.

*21. In $\triangle$ ABC, $\angle$ B $=40°$, $\angle$ C $=120°$. The bisector of $\angle$ B cuts AC at X and cuts the perpendicular from C to AB at N. Prove that XN = XC.

*22. In fig. 302, the sides of $\triangle$ ABC are produced as shown. If AB = AC, BG = BH, and AK = KG, find $\angle$ BAC and prove that HC = HK.

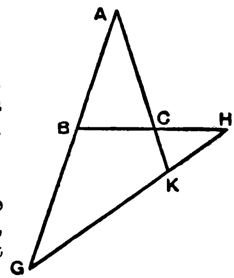


FIG. 302

## EXERCISE 34

[*Arrows indicate that lines are given parallel.*]

1. In fig. 303, AP bisects $\angle$BAC. Prove that AQ = QP.

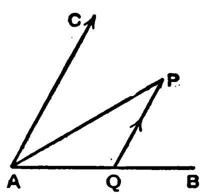


FIG. 303

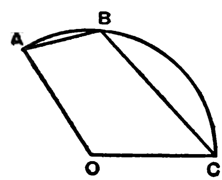


FIG. 304

2. In fig. 304, O is the centre of the circular arc ABC. Prove that $\angle$ OAB + $\angle$ OCB = $\angle$ ABC. [Join OB.]

[3] The side BC of $\triangle$ ABC is produced to D. If AB = AC, prove that $\angle$ ABD + $\angle$ ACD = 2 rt. $\angle$s.

4. The side BC of $\triangle$ ABC is produced to D. If $\angle$ ACD = 2$\angle$ BAC, prove that CA = CB.

[5] In $\triangle$ ABC, AB = AC. If the bisector of $\angle$ B meets AC at P, prove that $\angle$ APB = 3$\angle$ PBC.

6. In fig. 305, IB, IC are the bisectors of ∠ABC, ∠ACB. If AB = AC, prove that IB = IC.

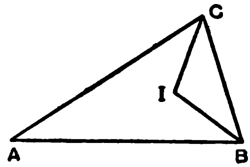


FIG. 305

7. ABCD is a quadrilateral in which AB = AD and ∠ABC = ∠ADC. Prove that CB = CD. [Join BD.]

[8] In △ABC, AB = AC. If a line parallel to BC cuts AB, AC at H, K, respectively, prove that BH = CK.

9. In fig. 306, AB = AC, BP = QC and BPQC is a straight line. Prove that ∠APQ = ∠AQP.

[Prove two triangles are congruent.]

[10] ABCD is a quadrilateral in which ∠B and ∠C are equal acute angles. If AB = CD, prove that ∠A = ∠D. [Produce BA and CD to meet at K.]

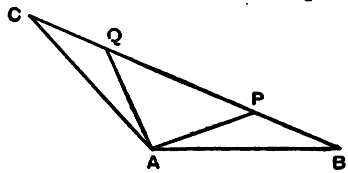


FIG. 306

[11] In △ABC, AB = AC. If AC is produced to any point X and if BX is joined, prove that ∠ABX + ∠AXB = 2∠ACB.

12. IB, IC are the bisectors of ∠ABC, ∠ACB of △ABC, see fig. 305. If AB = AC and if BC is produced to D, prove that ∠ACD = ∠BIC.

[13] D is the mid-point of the side BC of △ABC. If AD = BD, prove that ∠BAC is a right angle.

[14] In △ABC, AB = AC. BA is produced to E and AX is drawn bisecting ∠CAE. Prove that AX is parallel to BC.

[15] In △ABC, AB = AC. If CN is the perpendicular from C to AB, prove that ∠NCB = $\frac{1}{2}$∠A.

16. In fig. 307, P, Q, R are points on the sides of △ABC such that BR = PC and QC = PB. If AB = AC, prove that (i) PQ = PR, (ii) ∠RPQ = ∠B.

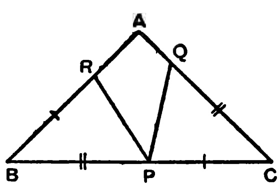


FIG. 307

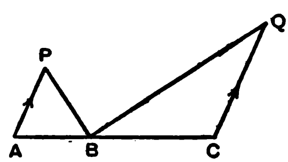


FIG. 308

17. In fig. 308, AP = AB, CB = CQ, and ABC is a straight line. Prove that ∠PBQ is a right angle.

**18.** In fig. 309, $\angle ABK = \angle C$ and $BN$ bisects $\angle KBC$. Prove that $AN = AB$.

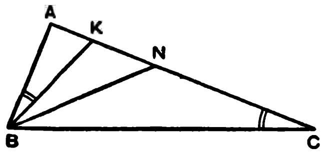



FIG. 309

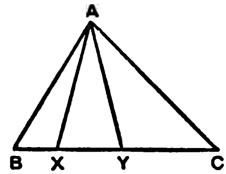



FIG. 310

**19.** In fig. 310, $\angle XAC = \angle B$ and $\angle YAB = \angle C$. Prove that $AX = AY$.

[20] In $\triangle ABC$, $AB = AC$; $BC$ is produced to any point $K$, and $AK$ is joined. If a triangle is drawn in which two of the angles are equal to $\angle CAK$, $\angle BAK$, respectively, prove that the third angle is equal to $2\angle CKA$.

**21.** In fig. 311, not drawn accurately, $AX = AY$ and the straight line $CYX$ bisects $\angle ACB$. Find an angle in the figure equal to $\angle ABC$.

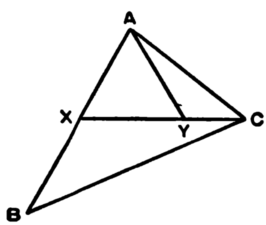



FIG. 311

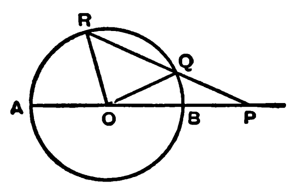



FIG. 312

[22] In fig. 312, $O$ is the centre of the circle; $PBOA$ and $PQR$ are straight lines. If $PQ$ is equal to the radius of the circle, prove that $\angle AOR = 3\angle BOQ$.

**23.** $ABC$ is an acute-angled triangle; $ABP$, $ACQ$ are equilateral triangles outside $\triangle ABC$. Prove that $\triangle PAC \equiv \triangle BAQ$.

If $CP$ cuts $BQ$ at $R$, prove that $\angle BRC = 120°$.

[24] $Y$ is any point on the side $BC$ of the equilateral triangle $ABC$; $BYK$ is an equilateral triangle outside $\triangle ABC$. Prove that (i) $AY = KC$; (ii) $\angle BAY = \angle KCY$; (iii) $\angle YAC = \angle YKC$.

25. In fig. 313, AB = AC and PQ = PC. Prove that ∠BCQ is a right angle.

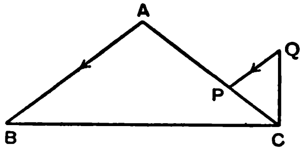


FIG. 313

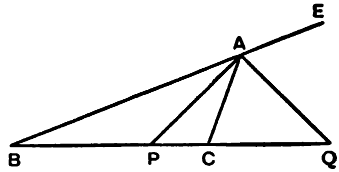


FIG. 314

[26] The side BA of △ABC is produced to E and the bisectors AP, AQ of ∠CAB, ∠CAE cut BC, BC produced, at P, Q, see fig. 314. If AP = AQ, prove that

(i) ∠APQ = 45°, (ii) ∠ACB − ∠ABC = 90°.

*27. In △ABC, ∠A is a right angle; AD is the perpendicular from A to BC. If P is a point on CB such that CP = CA, prove that AP bisects ∠BAD.

*28. In fig. 315, PBC, PKH are straight lines. If AH = AK, prove that $p = \frac{1}{2}(b - c)$.

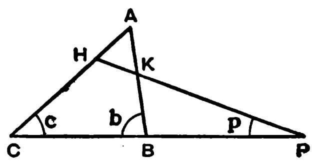


FIG. 315

*29. In △ABC, ∠B = ∠C = 40°; the bisector of ∠ABC meets AC at D; Q is a point on BC such that BQ = BD. Prove that CQ = AD. [Take R on BC so that BR = BA; join DR.]

*30. The side BC of △ABC is produced to D. The bisectors of ∠ABD, ∠ACD meet at P. If the line through P parallel to BC cuts BA, CA, produced if necessary, at Q, R respectively, prove that QR is equal to the difference of BQ and CR. [Prove PQ = QB; what line equals CR?]

*31. In fig. 316, AB = BC, CE = ED, and ACD, ABPE are straight lines. Prove that (i) △DPE ≡ △EBC; (ii) AB = PE.

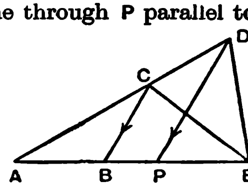


FIG. 316

*32. A line AD meets BC at D and divides △ABC into two isosceles triangles. Prove that in △ABC either one angle is a right angle or one angle is twice another angle or one angle is three times another angle.

*33. If, in fig. 312, K is a point on the circle such that the centre O lies inside △KQR. Prove that (i) ∠QOR = 2∠QKR, (ii) ∠QOR = 2∠QAR. [Join KO and produce it.]

## † THEOREM 13

**If two triangles have the THREE SIDES of one equal to the THREE SIDES of the other, each to each, the triangles are congruent.**

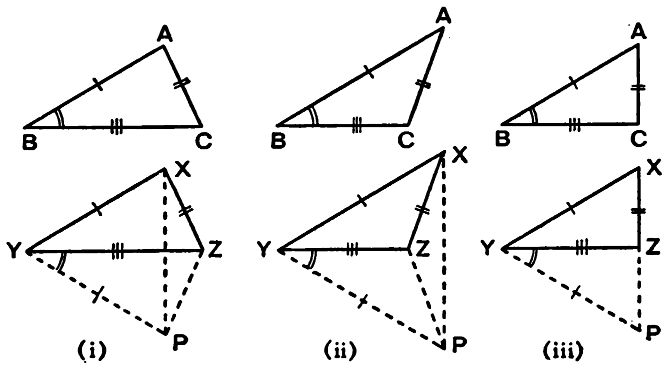



FIG. 317

**Given** two $\triangle$s ABC, XYZ, in which

$$AB = XY, \quad BC = YZ, \quad CA = ZX.$$

**To prove that** $\triangle$s ABC, XYZ are congruent.

**Construction.** On the opposite side of YZ from X, draw YP so that $\angle PYZ = \angle B$ and $YP = BA$. Join PZ, PX.

**Proof.** In $\triangle$s PYZ, ABC,

$$PY = AB \quad \textit{constr.},$$
$$YZ = BC \quad \textit{given},$$
$$\angle PYZ = \angle B \quad \textit{constr.},$$

$$\therefore \triangle\text{s } \begin{matrix}PYZ\\ABC\end{matrix} \text{ are congruent} \qquad SAS.$$

$$\therefore ZP = CA$$
and $\angle YPZ = \angle A$.

But $CA = ZX$ *given*,
$\therefore ZP = ZX$.

Also $YP = BA$ *constr.*,
and $BA = YX$ *given*,
$\therefore YP = YX$.

$\therefore$ YPX and ZPX are isosceles triangles,

$\therefore \angle YPX = \angle YXP$ and $\angle ZPX = \angle ZXP$ *base* $\angle$*s*,

$\therefore$ adding in fig. (i) and subtracting in fig. (ii),

$$\angle YPZ = \angle YXZ.$$

This is also true in fig. (iii) where XZP is a straight line.

But $\angle YPZ = \angle A$ *proved*,
$\therefore \angle YXZ = \angle A$.

$\therefore$ in $\triangle$s ABC, XYZ,

$AB = XY$ *given*,
$AC = XZ$ *given*,
$\angle A = \angle YXZ$ *proved*,

$\therefore \triangle$s $\begin{matrix} ABC \\ XYZ \end{matrix}$ are congruent SAS.

*Abbreviation for reference:* SSS *or* 3 sides.

NOTE. It may appear at first sight that cases (ii), (iii) can be avoided by choosing YZ and BC as two equal sides which are not shorter than the other pairs of equal sides. But actually the proof that in fact this leads only to case (i) is not easy and is certainly much longer than the consideration of the three cases.

† The proofs of Theorems 13, 14 are included in the text because it may be considered useful to take them as examples for oral discussion. Pupils should not be required to reproduce these proofs at a first reading.

## THEOREM 14

**If two triangles have TWO SIDES of the one equal to TWO SIDES of the other, each to each, and the angles opposite to one pair of equal sides are RIGHT ANGLES, the triangles are congruent.**

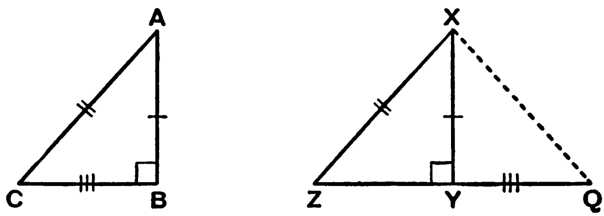


FIG. 318

**Given** two $\triangle$s ABC, XYZ, in which

$$AC = XZ, \ AB = XY, \ \angle B = \angle Y = 1 \text{ rt. } \angle,$$

**To prove that** $\triangle$s ABC, XYZ are congruent.

**Construction.** Produce ZY to Q, making $YQ = BC$. Join XQ.

**Proof.** Since $\angle XYZ = 1$ rt. $\angle$ and ZYQ is a straight line,

$$\angle XYQ = 1 \text{ rt. } \angle.$$

$\therefore$ in the $\triangle$s ABC, XYQ,

| | |
|---|---|
| $AB = XY$ | *given,* |
| $BC = YQ$ | *constr.,* |
| $\angle B = \angle XYQ$ | *rt.* $\angle$*s,* |

$\therefore$ $\triangle$s $\begin{matrix} ABC \\ XYQ \end{matrix}$ are congruent     SAS.

$\therefore$ $\angle C = \angle Q$ and $AC = XQ$;

but $AC = XZ$     *given,*   $\therefore$ $XZ = XQ$.

$\therefore$ $\angle Z = \angle Q$     *base* $\angle$*s, isos.* $\triangle$;

but $\angle C = \angle Q$     *proved,*   $\therefore$ $\angle C = \angle Z$.

$\therefore$ in the $\triangle$s ABC, XYZ,

| | |
|---|---|
| $AB = XY$ | *given,* |
| $\angle B = \angle XYZ$ | *rt. $\angle$s, given,* |
| $\angle C = \angle Z$ | *proved,* |

$\therefore$ $\triangle$s $\begin{matrix}ABC\\XYZ\end{matrix}$ are congruent AAS.

*Abbreviation for reference:* RHS *or* rt. $\angle$, hyp., side.

**The Ambiguous Case.** The size and shape of a triangle are not necessarily fixed when *the lengths of two sides* and the size of an angle *opposite* to one of these sides, *i.e.* a *not-included angle,* are given, unless the given angle is a right angle (p. 56).

**ABC, XYZ are two triangles such that**

$$AB = XY, \quad AC = XZ, \quad \angle B = \angle Y,$$

**then *either* $\angle C = \angle Z$ and the triangles are congruent,**

***or* $\angle C$, $\angle Z$ are supplementary.**

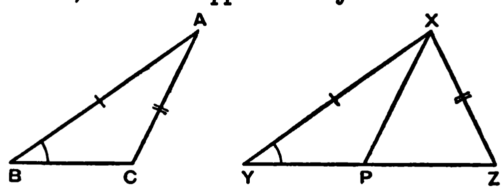



FIG. 319

If $BC = YZ$, $\triangle$s $\begin{matrix}ABC\\XYZ\end{matrix}$ are congruent SAS.

$\therefore \angle C = \angle Z.$

If BC is not equal to YZ, one of them must be the greater; suppose YZ is the greater and cut off a part YP equal to BC.

Then $\triangle$s $\begin{matrix}ABC\\XYP\end{matrix}$ are congruent SAS.

$\therefore \angle C = \angle XPY$ and $AC = XP$.

But $AC = XZ$ *given,* $\therefore XP = XZ$.

$\therefore \angle XPZ = \angle Z$ *base $\angle$s, isos. $\triangle$,*

But $\angle XPY + \angle XPZ = 2$ rt. $\angle$s *adj. $\angle$s on st. line,*

$\therefore \angle C + \angle Z = 2$ rt. $\angle$s.

## EXERCISE 35

1. AB and CD are two equal chords of a circle, centre O; prove that $\angle AOB = \angle COD$.

[2] The sides of the quadrilateral ABCD are all equal; prove that AC bisects $\angle BAD$.

3. In fig. 320, BA = BC and KA = KC; prove that $\angle BAK = \angle BCK$.

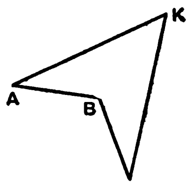



FIG. 320

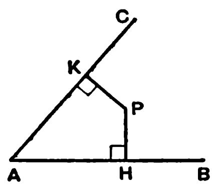



FIG. 321

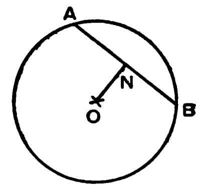



FIG. 322

4. In fig. 321, the perpendiculars from P to AB, AC are equal; prove that AP bisects $\angle BAC$.

5. In fig. 322, if ON is the perpendicular from the centre O of the circle to a chord AB, prove that AN = NB.

[6] In fig. 322, if N is the mid-point of a chord AB of a circle, centre O, prove that ON is perpendicular to AB.

7. ABCD is a quadrilateral such that AB = CD and AD = BC. Prove that AD is parallel to BC. [Join AC.]

[8] Two circles cut at A, B; O is the centre of one circle; prove that the centre of the other circle lies on the bisector of $\angle AOB$.

9. N is the mid-point of the side BC of $\triangle ABC$; NX, NY are the perpendiculars from N to AB, AC. If NX = NY, prove that AB = AC.

[10] ABCD is a quadrilateral such that AD = BC and AC = BD. If AC cuts BD at K, prove that (i) KC = KD, (ii) AB || DC.

11. AB, CD are two equal lines not in the same plane. If AC = BD, prove that $\angle BAC = \angle BDC$. If the triangle ABC is rotated about BC, must A pass through D?

12. In $\triangle ABC$, AB = AC and $\angle A = 1$ rt. $\angle$; XAY is any straight line through A; BH, CK are the perpendiculars from B, C to XAY; prove that AH = CK.

13. In fig. 323, $AB = BH$, $BC = BK$, $AK = CH$, and $ABC$, $BHK$ are straight lines. Prove that (i) $\angle ABK = 1$ rt. $\angle$, (ii) $CH$ is perpendicular to $AK$.

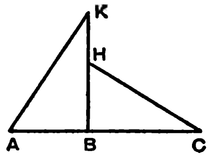



Fig. 323

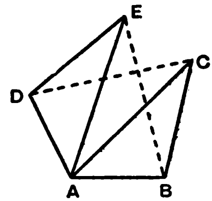



Fig. 324

*14. In fig. 324, $AB = AD$, $AC = AE$, $BC = DE$; prove that $BE = CD$.

*15. In fig. 325, the bisector of $\angle BAC$ meets at $P$ the line which bisects $BC$ at right angles; $PX$, $PY$ are the perpendiculars from $P$ to $AB$, $AC$.

Find in the figure three pairs of congruent triangles and prove the congruence. Prove also that $AX = \frac{1}{2}(AB + AC)$.

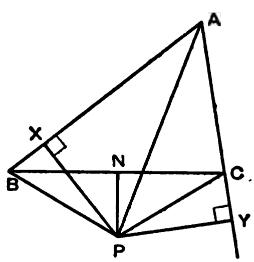



Fig. 325

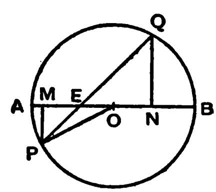



Fig. 326

*16. In fig. 326, $AB$ is the diameter of the circle, centre $O$; the chord $PQ$ makes 45° with $AB$; $PM$, $QN$ are the perpendiculars from $P$, $Q$ to $AB$. Prove $ME = ON$. [Prove $\triangle PMO \equiv \triangle ONQ$.]

*17. $P$ is any point on the side $AB$ of a triangle $ABC$. The triangle is rotated about $BC$; $A'$, $P'$ are the new positions of $A$, $P$. Prove that $\angle P'AC = \angle PA'C$.

*18. $P$, $Q$ are two points on the perpendicular bisector of $AB$. At $P$ a perpendicular $PR$ is erected to the plane of $PQ$ and $AB$. Prove that $\angle ARQ = \angle BRQ$.

**Formal Constructions.** In theoretical constructions the instruments allowed are restricted to—

(i) a straightedge (ungraduated ruler) for

(*a*) joining two given points by a straight line,

(*b*) producing a given straight line;

(ii) a compass for

(*a*) drawing a circle with a given centre and radius,

(*b*) cutting off from a straight line a length equal to that of a given straight line.

When performing a construction, the figure must be drawn accurately and all construction-lines must be shown; **none of them must be rubbed out.**

To secure **a high standard of accuracy** in constructions,

(i) use a hard pencil and see that it is properly sharpened.

A line drawn on paper has width, but the less the width, the more accurate your drawing will be. It is impossible to draw a fine line with a soft pencil.

(ii) Draw circles with long radii, so that points which have to be joined are as far apart as possible.

There is less error in the position of a line which is obtained by joining two points far apart than two points close together. You may see why this is so if you try to hold a long rod at two points close to one end when someone else is pushing at the other end at right angles to the rod.

(iii) Avoid intersections of lines and circles in which the angles at which they cut are small.

Think how an intersection would appear under a microscope by regarding lines as strips, say 1 cm. wide. If the angle formed by the strips is small, the area common to the two strips is large.

(iv) Take care of your compass. You will spoil it if you use it for ordinary writing or for ruling lines.

## CONSTRUCTION 1

**Bisect a given angle.**

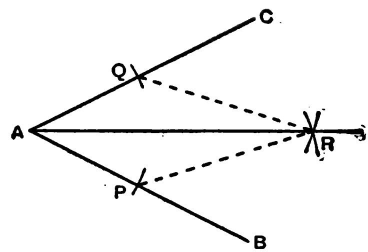



FIG. 327

**Given** an angle BAC.

**To construct** the line bisecting $\angle$ BAC.

**Construction.** With centre A and any radius, draw an arc of a circle cutting AB, AC at P, Q.
With centres P, Q and with any sufficient radius, the same for each, draw arcs of circles cutting at R.
Join AR.
Then AR is the required bisector.

**Proof.** Join PR, QR.

In $\triangle$s PAR, QAR,

AP = AQ *radii of the same circle,*
PR = QR *radii of equal circles,*
AR = AR.

$\therefore$ $\triangle$s $\begin{matrix} PAR \\ QAR \end{matrix}$ are congruent SSS,

$\therefore$ $\angle PAR = \angle QAR$.

$\therefore$ AR bisects $\angle$ BAC.

## CONSTRUCTION 2

**Bisect a given finite straight line.**

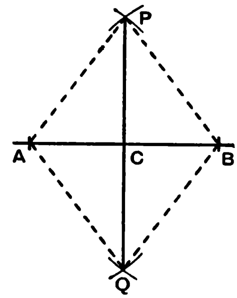


FIG. 328

**Given** a finite line AB.

**To construct** the mid-point of AB.

**Construction.** With centres A, B and any sufficient radius, the same for each, draw arcs of circles cutting at P, Q.
Join PQ and let it cut AB at C.
Then C is the mid-point of AB.

**Proof.** Join PA, PB, QA, QB.

In $\triangle$s PAQ, PBQ,

AP = BP *radii of equal circles,*
AQ = BQ *radii of equal circles,*
PQ = PQ.

$\therefore$ $\triangle$s $\begin{matrix}PAQ\\PBQ\end{matrix}$ are congruent SSS.

$\therefore$ $\angle$APQ = $\angle$BPQ.

$\therefore$ in $\triangle$s PAC, PBC,

AP = BP *radii of equal circles,*
PC = PC
$\angle$APC = $\angle$BPC *proved.*

$\therefore$ $\triangle$s $\begin{matrix}PAC\\PBC\end{matrix}$ are congruent SAS.

$\therefore$ AC = BC.

NOTE. Since also $\angle$ ACP = $\angle$ BCP and since these are adjacent angles on a straight line, each of these angles is a right angle. Therefore PQ is the perpendicular bisector of AB.

## CONSTRUCTION 3

**Draw a straight line at right angles to a given straight line from a given point in the line.**

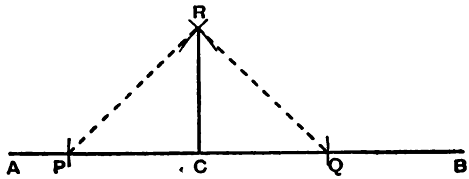



FIG. 329

**Given** a point C in a line AB.

**To construct** a line from C perpendicular to AB.

**Construction.** With centre C and any radius, draw an arc of a circle cutting AB at P, Q.
With centres P, Q and any sufficient radius, the same for each, draw arcs of circles cutting at R.
Join CR.
Then CR is the required line perpendicular to AB.

**Proof.** Join PR, QR.

In $\triangle$s PCR, QCR,

CP = CQ *radii of the same circle,*
PR = QR *radii of equal circles,*
CR = CR.

$\therefore$ $\triangle$s $\begin{matrix}\text{PCR}\\ \text{QCR}\end{matrix}$ are congruent SSS.

$\therefore$ $\angle$PCR = $\angle$QCR,

but these are adjacent angles on a straight line,

$\therefore$ $\angle$PCR is a right angle.

NOTE. This construction is a special case of the construction for bisecting a given angle. In fig. 329, CR bisects the straight angle ACB.

Two alternative methods for Construction 3 are indicated in figs. 330, 331. It is a useful exercise for the reader to perform each of these constructions and prove that they are correct.

**Given a point C on AB, to draw the line through C perpendicular to AB.**

(1) With centre C and any radius, draw the arc PQR cutting CB at P.

With centre P and the same radius, draw an arc cutting the arc PQR at Q.

With centre Q and the same radius, draw an arc RS cutting the arc PQR at R.

With centre R and the same radius, draw an arc cutting the arc RS at S.

Join CS.

Then CS is perpendicular to AB.

[For the proof, note that $\triangle$s PCQ, CQR, SQR are equilateral.]

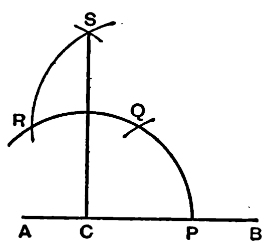



Fig. 330

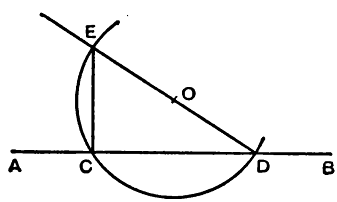



Fig. 331

(2) Take any point O outside AB.

Draw a circle, centre O, radius OC, and let it cut AB again at D.

Join DO and produce it to meet the circle at E.

Join CE.

Then CE is perpendicular to AB.

[For the proof, note that $\triangle$s ODC, OEC are isosceles.]

## CONSTRUCTION 4

**Draw a perpendicular to a given straight line of unlimited length from a given point outside the line.**

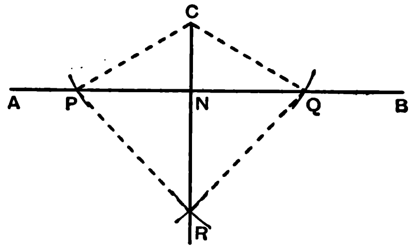


FIG. 332

**Given** a line AB and a point C outside AB.

**To construct** the line from C perpendicular to AB.

**Construction.** With C as centre and any sufficient radius, draw an arc of a circle cutting AB at P, Q.

With P, Q as centres and any sufficient radius, the same for each, draw arcs of circles cutting at R.

Join CR and let it cut AB at N.

Then CN is the required perpendicular from C to AB.

**Proof.** Join CP, CQ, RP, RQ.

In $\triangle$s CPR, CQR,

CP = CQ *radii of the same circle,*
PR = QR *radii of equal circles,*
CR = CR.

$\therefore$ $\triangle$s $\begin{matrix}\text{CPR}\\\text{CQR}\end{matrix}$ are congruent SSS,

$\therefore$ $\angle$PCR = $\angle$QCR.

$\therefore$ in the $\triangle$s CPN, CQN,

CP = CQ *radii of the same circle,*
CN = CN
$\angle$PCN = $\angle$QCN *proved.*

$\therefore$ $\triangle$s $\begin{matrix}\text{CPN}\\\text{CQN}\end{matrix}$ are congruent SAS.

$\therefore$ $\angle$CNP = $\angle$CNQ,

but these are adj. $\angle$s on st. line,

$\therefore$ each is a right angle.

Two alternative methods for Construction 4 are indicated in figs. 333, 334. It is a useful exercise for the reader to perform each of these constructions and prove that they are correct.

**Given a line AB and a point C outside AB, to draw the line from C perpendicular to AB.**

(1) Take any two points P, Q on AB.

Draw an arc of a circle, centre P, radius PC.

Draw an arc of a circle, centre Q, radius QC, cutting the first arc at R.

Join CR, cutting AB at N.

Then CN is the perpendicular from C to AB.

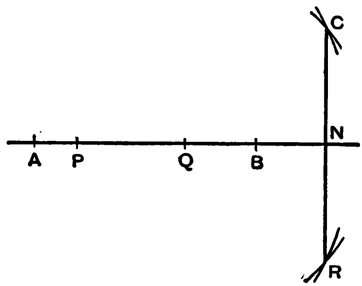



FIG. 333

[For the proof, prove $\triangle QPC \equiv \triangle QPR$ and then prove $\triangle NPC \equiv \triangle NPR$.]

NOTE. The points P and Q should be taken as far apart as possible because this makes it easier to fix the position of R accurately.

(2) Take any point D on AB.

Join CD and *construct* the mid-point O of CD.

With centre O, radius OC, draw an arc of a circle cutting AB at N.

Join CN.

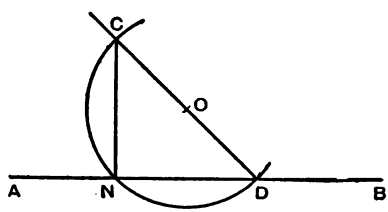



FIG. 334

Then CN is the perpendicular from C to AB.

[For the proof, note that $\triangle$s ODN, OCN are isosceles.]

The next construction shows how an angle can be copied. The method consists in constructing a triangle one of whose angles is the given angle and then constructing a triangle the lengths of whose sides are equal to those of the first triangle.

## CONSTRUCTION 5

**From a given point in a given straight line, draw a straight line making with the given line an angle equal to a given angle.**

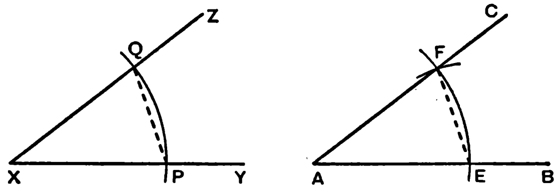



FIG. 335

**Given** a point A on a given line AB and an angle YXZ.

**To construct** a line AC such that $\angle CAB = \angle ZXY$.

**Construction.** With centre X and any radius, draw an arc of a circle cutting XY, XZ at P, Q.

With centre A and the same radius, draw an arc EF of a circle cutting AB at E.

With centre E and radius equal to PQ, draw an arc of a circle cutting the arc EF at F.

Join AF and produce it to C.

Then AC is the required line.

**Proof.** Join PQ, EF.

In $\triangle$s PXQ, EAF,

| | |
|---|---|
| $XP = AE$ | *radii of equal circles,* |
| $XQ = AF$ | *radii of equal circles,* |
| $PQ = EF$ | *constr.* |

$\therefore \triangle\text{s } \begin{matrix} PXQ \\ EAF \end{matrix}$ are congruent SSS,

$\therefore \angle YXZ = \angle BAC$.

The most important use made of Construction 5 is for constructing a line passing through a given point parallel to a given straight line. See also Ex. 36, No. 14.

*In practical work,* parallels are always drawn by using set-squares. But if a *formal* construction is required, set-squares cannot be used.

## CONSTRUCTION 6

**Through a given point, draw a straight line parallel to a given straight line not passing through the given point.**

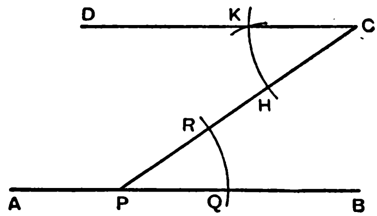


FIG. 336

**Given** a point C outside a given line AB.

**To construct** a line through C parallel to AB.

**Construction.** Take any point P on AB. Join PC.
From the point C on CP, construct the line CD so that $\angle$PCD is equal to $\angle$CPB and alternate to it.
Then CD is the required line.

**Proof.** Since the transversal PC cuts the lines AB, DC so as to make the alternate angles CPB, PCD equal, AB is parallel to DC.

## EXERCISE 36

[*Use only a compass and straightedge for these constructions.*]

1. Construct an equilateral triangle, given one side.

2. Construct an isosceles right-angled triangle, given one of the shorter sides.

3. Take two points A and B. Construct a point C on AB produced such that BC = 3AB.

Construct the following angles:—

4. 30°. 5. 45°. 6. 105°. [7] $22\frac{1}{2}$°. [8] 75°.

[9] Construct a right angle and construct two lines dividing the right angle into three equal angles.

10. Take two points A and D. Construct an equilateral triangle ABC such that AD is the perpendicular from A to BC.

11. Draw a triangle ABC and construct the bisector of each angle of the triangle. Do the bisectors meet at a point?

[12] Draw an obtuse angle. Divide it into four equal angles.

13. Take two points A and B. Construct C and D so that $\angle$ABC and $\angle$BCD are alternate angles and are right angles and so that AB = BC = CD. Do AD and BC bisect each other?

14. Draw any triangle ABC. Construct D so that CD = BA and AD = BC. Explain why this is a construction for drawing a line through A parallel to BC.

15. Draw any triangle and construct the perpendicular bisector of each side. Do they meet at a point?

[16] Draw any circle, centre O, and draw any chord PQ which is not a diameter. Construct the perpendicular bisector of PQ. Does it pass through O?

17. Draw a circle and take three points A, B, C on the circumference. Construct the bisector of $\angle$BAC and the perpendicular bisector of BC; do they meet, when produced, on the circumference?

18. Draw any acute-angled triangle and construct the perpendicular from each vertex to the opposite side.

[19] Draw a triangle ABC so that $\angle$C is obtuse. Construct the perpendiculars from A to BC produced, from B to AC produced, and from C to AB. Do these lines, when produced, meet at a point?

20. Take three points P, Q, R so that $\angle$PQR is obtuse. Join PQ and QR. Construct a line through R perpendicular to PQ *without producing* PQ.

21. Draw a triangle ABC and take a point D inside the triangle. Construct a point P on BC such that $\angle$DPC = $\angle$B.

[22] Draw a triangle ABC and a straight line DE. Construct a line DF such that $\angle$EDF = $\angle$BAC.

*23. Draw a triangle ABC in which AB is greater than AC. Produce AB, AC to P, Q. Construct the bisectors of $\angle$ABC, $\angle$ACB and let them meet at H. Construct the bisectors of $\angle$PBC, $\angle$QCB and let them meet at K. Does the bisector of $\angle$BAC pass through H and K?

*24. Draw a triangle ABC in which AB is greater than AC. Construct a point P on BC such that the perpendiculars from P to AB and AC are equal. State your method shortly.

*25. Draw a triangle ABC in which AB is greater than AC. Construct a line AD in the angle BAC such that the perpendiculars from B and C to AD are equal. State your method shortly.

*26. Draw a triangle ABC and a straight line XY. Construct P on XY such that AP makes with XY an angle equal to $\angle$B.

*27. Draw an acute angle ACB and a straight line DE. Construct points P, Q on CA, CB such that $\angle$CPQ is a right angle and PQ = DE. State your method shortly.

*28. Draw a straight line AB and take any two points C, D on the same side of AB. Construct P on AB so that $\angle$APC = $\angle$BPD. [Find a point D′ such that AB is the perpendicular bisector of DD′; join CD′.]

## Parallelograms

**Definition.** A quadrilateral which has both pairs of opposite sides parallel, see fig. 337, is called a **parallelogram.**

*Abbreviation:* ||gram *or* parm.

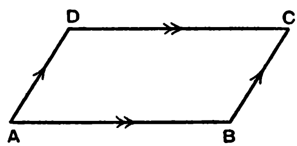


Fig. 337

It is suggested that the properties of a parallelogram in Theorems 15, 16 and the tests for a quadrilateral to be a parallelogram in Theorems 17–20 should be taken orally as riders, as follows:—

## Examples for Oral Discussion

### Properties of a Parallelogram

1. ABCD is a parallelogram. **Prove that**

(i) AB = DC, AD = BC.
(ii) $\angle$A = $\angle$C, $\angle$B = $\angle$D.

*Draw your own figure. Show on it by suitable markings what is given.* Join BD.

Explain with full reasons why $\triangle$ABD $\equiv$ $\triangle$CDB.

Note. Your proof also shows that the area of a parallelogram is bisected by a diagonal.

**2. If the diagonals of a parallelogram ABCD cut at K, prove that**

AK = KC and BK = KD.

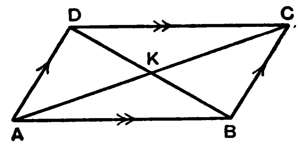



FIG. 338

Explain with full reasons why △AKB ≡ △CKD.

**Tests for a Parallelogram**

**3. ABCD is a quadrilateral in which AB = DC and AB is parallel to DC. Prove that ABCD is a parallelogram.**

*Draw your own figure and show on it by suitable markings what is given.* Join BD.

Why is it sufficient to prove that AD is parallel to BC?

Explain with full reasons why △ABD ≡ △CDB.

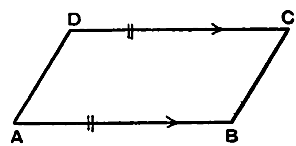



FIG. 339

**4. ABCD is a quadrilateral in which ∠A = ∠C and ∠B = ∠D. Prove that ABCD is a parallelogram.**

*Draw your own figure and show on it by suitable markings what is given. No construction.*

What is the sum of the angles of any quadrilateral?

Use the data to prove that AB is parallel to DC and that AD is parallel to BC.

**5. ABCD is a quadrilateral in which AB = DC and AD = BC. Prove that ABCD is a parallelogram.**

*Draw your own figure and show on it by suitable markings what is given.* Join BD.

Explain with full reasons why △ABD ≡ △CDB.

Complete the proof, *i.e.* prove that AB is parallel to DC, and AD is parallel to BC.

6. ABCD is a quadrilateral in which the diagonals bisect each other at K, i.e. AK = KC and BK = KD. Prove that ABCD is a parallelogram.

Use △s AKB, CKD to prove that AB is parallel to DC.

How can you prove that AD is parallel to BC?

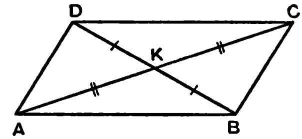


Fig. 340

7 (*A discovery example.*) Use the results of the preceding examples to discover as many properties as you can of fig. 341 in which ABCD and ABEF are given parallelograms. Give reasons.

Answer the same question if ABCD, ABEF are parallelograms in different planes. [Assume that two lines which are parallel to the same line are parallel to one another, even if all three lines do not lie in one plane.]

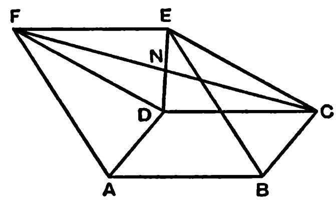
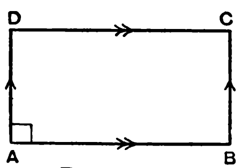


Fig. 341

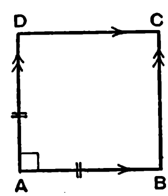
## Rectangle, Square, and Rhombus

A parallelogram in which ONE angle is a right angle is called a **rectangle** (*abbreviation*, rect.).



Rectangle

Fig. 342



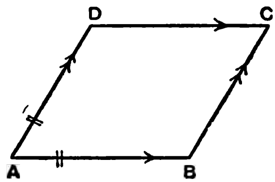
Square

Fig. 343

A rectangle which has TWO ADJACENT sides equal is called a **square** (*abbreviation*, sq.).

The square ABCD is often called *the square on* AB, or *sq. on* AB.

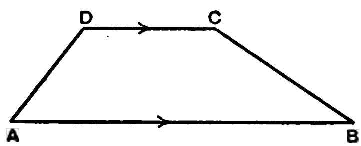
A parallelogram which has TWO ADJACENT sides equal is called a **rhombus.**




RHOMBUS

FIG. 344

A quadrilateral which has one, and only one, pair of sides parallel is called a **trapezium.** If the other two sides are equal, the trapezium is called **isosceles.** Thus in fig. 345, if $AB \parallel DC$ and if $AD = BC$, and if $AD$ is not parallel to $BC$, $ABCD$ is an isosceles trapezium.




TRAPEZIUM

FIG. 345

These definitions must be noted carefully. A definition must include sufficient to be free from ambiguity, but must not include facts that can be deduced from the definition. For example, a rectangle is *defined* as a parallelogram in which **ONE** angle is a right angle; it can then be **proved** that the remaining three angles are also right angles; this fact is deduced from the definition, it is wrong to include it in the definition.

The reader should use the properties of a parallelogram and the definitions on pp. 144, 146 to obtain the following important results *which may be assumed in rider-work unless a proof is explicitly asked for.*

**I. Properties of a Rectangle**

(i) **All the angles of a rectangle are right angles.**

If $\angle A = 1$ rt. $\angle$,

then

$\angle B = \angle C = \angle D = 1$ rt. $\angle$.

[*No construction:* use parallels.]

(ii) **The diagonals of a rectangle are equal.**

[Prove that $\triangle DAB \equiv \triangle CBA$.]

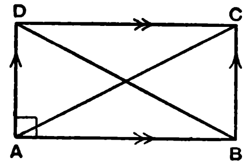


FIG. 346

(iii) **If the diagonals of a parallelogram are equal, the parallelogram is a rectangle.**

[Prove that $\triangle DAB \equiv \triangle CBA$; why does $\angle A + \angle B = 2$ rt. $\angle$s?]

**II. Properties of a Square**

(i) **All the sides of a square are equal.**

If $AB = AD$,

then $AB = BC = CD$.

[*No construction:* opp. sides ‖gram are equal.]

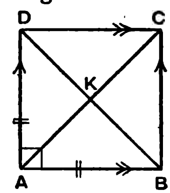


FIG. 347

(ii) **The diagonals of a square are equal and cut at right angles.**

(iii) **The angle which a diagonal makes with a side of the square is 45°.**

(iv) **If the diagonals of a parallelogram are equal and cut at right angles, the parallelogram is a square.**

**III. Properties of a Rhombus**

(i) **All the sides of a rhombus are equal.**

If $AB = AD$,

then $AB = BC = CD$.

(ii) **The angles of a rhombus are bisected by the diagonals.**

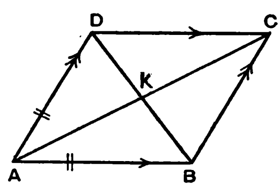


FIG. 348

(iii) **The diagonals of a rhombus cut at right angles.**

(iv) **If the diagonals of a parallelogram cut at right angles, the parallelogram is a rhombus.**

# NUMERICAL EXAMPLES

## EXERCISE 37

1. ABCD is a rectangle. If $\angle BAC = 32°$, find $\angle DBC$.

[2] ABCD is a rectangle. If $\angle ACD = 67°$, find $\angle ADB$.

3. ABCD is a rhombus. If $\angle ABC = 56°$, find $\angle ACD$.

[4] ABCD is a rhombus. If $\angle BAC = 35°$, find $\angle ADC$.

5. The diagonals of a rectangle ABCD cut at K. If $\angle AKB = 110°$, find $\angle ACB$ and $\angle ACD$.

[6] The diagonals of a rectangle ABCD cut at N. If $\angle ABD = 74°$, find $\angle DNC$.

7. ABCD is a square; a straight line CPQ cuts BD at P and cuts BA at Q. If $\angle CPD = 80°$, find $\angle CQA$.

8. ABCDE is a regular pentagon; ABPQ is a square inside the pentagon. Find $\angle CBP$ and $\angle DBQ$.

[9] ABC is an equilateral triangle; BCPQ and BCHK are the two squares on BC. Find $\angle APB$ and $\angle AHB$.

10. The diagonals of a square ABCD cut at K. From AB a part AQ is cut off equal to AK. Prove that $\angle AKQ = 3\angle BKQ$.

[11] The side AD of the square ABCD is produced to E, and the bisector of $\angle EDB$ meets AC produced at X. Find $\angle AXD$ and prove that $CD = CX$.

12. ABCD is a rhombus in which $\angle B = 108°$; CAPQ is another rhombus such that P lies on AB produced. Find the acute angle which AQ makes with BC.

[13] ABCDE is a regular pentagon; ABP is an equilateral triangle inside the pentagon. Find the angles of $\triangle PEB$.

14. The diagonals of a rectangle ABCD cut at K; KAP is an equilateral triangle drawn so that B and P are on the same side of AC. If $\angle ACD = 25°$, find the angles of $\triangle APB$.

*15. ABCD is a square; ABX is an equilateral triangle inside the square. Find $\angle DXC$.

*16. ABCDE is a regular pentagon and EDCP is a parallelogram inside the pentagon. Find $\angle EPA$ and prove that APC is a straight line.

*17. The diagonals of the rectangle ABCD cut at K, and AK is greater than AB. The circle, centre A, radius AK, cuts AB produced at E. If $\angle AKB = 4\angle BKE$, find $\angle BAC$.

*18. ABCD is a square; P is a point on CA produced such that the parallelogram DAPQ is a rhombus. If QC cuts PD at R, find the angles of $\triangle DRQ$ and prove that $RP = RC$.

## THEOREM 15

**(1) The opposite sides and angles of a parallelogram are equal.**

**(2) Each diagonal bisects the area of the parallelogram.**

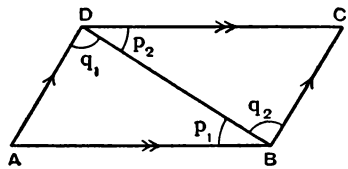


FIG. 349

**Given** a parallelogram ABCD and a diagonal BD,

**To prove that** (1) AB = DC, AD = BC

and $\angle$A = $\angle$C, $\angle$B = $\angle$D.

(2) Area of $\triangle$ABD = Area of $\triangle$CBD.

**Proof.** (1) With the notation in the figure,

in $\triangle$s ABD, CDB,

$p_1 = p_2$ *alt.* $\angle$*s*, AB || DC,

$q_1 = q_2$ *alt.* $\angle$*s*, AD || BC,

BD = DB.

$\therefore$ $\triangle$s $\begin{matrix}\text{ABD}\\ \text{CDB}\end{matrix}$ are congruent ASA.

$\therefore$ AD = CB, AB = CD,

and $\angle$A = $\angle$C.

(2) Since $\triangle$ABD $\equiv$ $\triangle$CDB, the triangles are equal in area.

Similarly, by joining AC it may be proved

that $\triangle$ABC $\equiv$ $\triangle$CDA.

$\therefore$ $\angle$B = $\angle$D,

and AC bisects the area of the parallelogram.

*Abbreviations for reference:* (1) opp. sides ||gram.
(2) opp. $\angle$s ||gram.

## THEOREM 16

**The diagonals of a parallelogram bisect one another.**

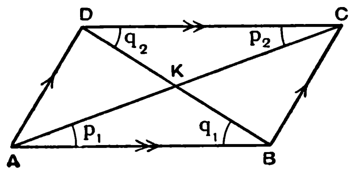



FIG. 350

**Given** a parallelogram ABCD whose diagonals AC, BD cut at K.

**To prove that** AK = KC
and BK = KD.

**Proof.** With the notation in the figure,

in $\triangle$s AKB, CKD,

| | |
|---|---|
| $p_1 = p_2$ | *alt.* $\angle$*s*, AB ∥ DC, |
| $q_1 = q_2$ | *alt.* $\angle$*s*, AB ∥ DC, |
| AB = CD | *opp. sides* ∥*gram.* |

$\therefore$ $\triangle$s $\begin{matrix}\text{AKB}\\ \text{CKD}\end{matrix}$ are congruent ASA.

$\therefore$ BK = DK
and AK = CK.

*Abbreviation for reference:* diags. ∥gram.

## THEOREM 17

**If one pair of opposite sides of a quadrilateral are equal and parallel, the other pair of opposite sides are also equal and parallel.**

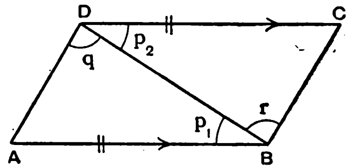



FIG. 351

**Given** a quadrilateral ABCD in which

AB = DC and AB ∥ DC.

**To prove that** AD = BC and AD ∥ BC.

**Construction.** Join BD.

**Proof.** With the notation in the figure,

in △s ABD, CDB,

AB = CD *given*,
BD = DB,
$p_1 = p_2$ *alt.* ∠*s*, AB ∥ DC.

∴ △s $\begin{matrix}\text{ABD}\\ \text{CDB}\end{matrix}$ are congruent SAS.

∴ AD = CB,

and $q = r$,

but these are alt. ∠s,

∴ AD is parallel to BC.

*Abbreviation for reference:* 2 sides equal and ∥.

This theorem is also stated in the form:

**A quadrilateral, which has one pair of equal and parallel sides, is a parallelogram.**

### THEOREM 18

**If the opposite angles of a quadrilateral are equal, the quadrilateral is a parallelogram.**

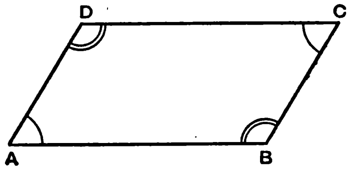


FIG. 352

**Given** a quadrilateral ABCD in which

$$\angle A = \angle C \quad \text{and} \quad \angle B = \angle D.$$

**To prove that** ABCD is a parallelogram.

**Proof.** The sum of the angles of a quadrilateral is 4 rt. $\angle$s,

$$\therefore \angle A + \angle B + \angle C + \angle D = 4 \text{ rt. } \angle\text{s},$$

but $\angle A = \angle C$ and $\angle B = \angle D$ *given*,

$$\therefore 2\angle A + 2\angle B = 4 \text{ rt. } \angle\text{s},$$

$$\therefore \angle A + \angle B = 2 \text{ rt. } \angle\text{s};$$

but these are interior angles on the same side of the transversal AB,

∴ AD is parallel to BC.

In the same way it may be proved that

$$\angle A + \angle D = 2 \text{ rt. } \angle\text{s};$$

but these are interior angles on the same side of the transversal AD,

∴ AB is parallel to DC.

∴ both pairs of opposite sides of ABCD are parallel;

∴ ABCD is a parallelogram.

*Abbreviation for reference:* opp. $\angle$s equal.

## THEOREM 19

**If the opposite sides of a quadrilateral are equal, the quadrilateral is a parallelogram.**

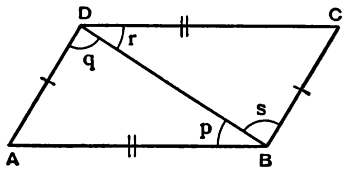



FIG. 353

**Given** a quadrilateral ABCD in which

AB = DC and AD = BC.

**To prove that** ABCD is a parallelogram.

**Construction.** Join BD.

**Proof.** In $\triangle$s BAD, DCB,

AB = CD *given*,
AD = CB *given*,
BD = DB.

$\therefore$ $\triangle$s $\begin{matrix}\text{BAD}\\ \text{DCB}\end{matrix}$ are congruent SSS.

$\therefore$ with the notation in the figure,

$$p = r,$$

but these are alt. $\angle$s,

$\therefore$ AB is parallel to DC;

and $q = s$,

but these are alt. $\angle$s,

$\therefore$ AD is parallel to BC.

$\therefore$ both pairs of opposite sides of ABCD are parallel;

$\therefore$ ABCD is a parallelogram.

*Abbreviation for reference:* opp. sides equal.

## THEOREM 20

**If the diagonals of a quadrilateral bisect each other, the quadrilateral is a parallelogram.**

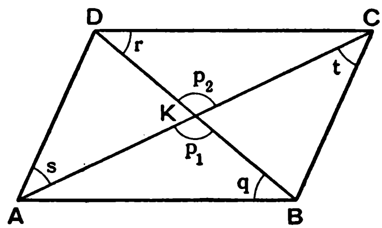



FIG. 354

**Given** a quadrilateral ABCD in which the diagonals AC, BD bisect each other at K, so that

$$AK = KC \quad \text{and} \quad BK = KD.$$

**To prove that** ABCD is a parallelogram.

**Proof.** With the notation in the figure, in $\triangle$s AKB, CKD,

| | |
|---|---|
| $AK = CK$ | *given,* |
| $BK = DK$ | *given,* |
| $p_1 = p_2$ | *vert. opp.* $\angle$*s.* |

$\therefore$ $\triangle$s $\begin{matrix} AKB \\ CKD \end{matrix}$ are congruent SAS.

$\therefore$ $q = r$;

but these are alt. $\angle$s,

$\therefore$ AB is parallel to DC.

Similarly, from the $\triangle$s AKD, CKB it may be proved that the alt. $\angle$s $s$, $t$ are equal,

$\therefore$ AD is parallel to BC.

$\therefore$ both pairs of opposite sides of ABCD are parallel;

$\therefore$ ABCD is a parallelogram.

*Abbreviation for reference:* diags. bisect each other.

## EXERCISE 38

1. State, without proof, what you know about the parallelogram ABCD (i) if AC bisects $\angle$BAD, (ii) if AC = BD, (iii) if AC is perpendicular to BD.

2. In fig. 355, ABCD and ABXY are parallelograms such that DCYX is a straight line. Use the SAA test to prove that $\triangle$ADY $\equiv$ $\triangle$BCX.

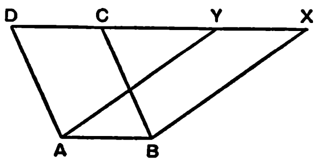


Fig. 355

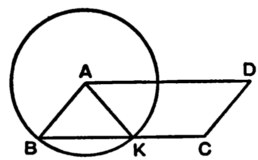


Fig. 356

3. In fig. 356, ABCD is a parallelogram and A is the centre of the circle. Prove that $\angle$KAD = $\angle$CDA. [No construction.]

4. The diagonals of the parallelogram ABCD cut at K; any line through K cuts AB, CD at X, Y. Prove that KX = KY.

[5] P is the mid-point of the side BC of a parallelogram ABCD; DP and AB meet, when produced, at Q. Prove that AB = BQ.

[6] ABCD is a parallelogram. Prove that the perpendiculars from B and D to AC are equal.

7. In fig. 357, ABCD and ABPQ are parallelograms. Prove that CDQP is a parallelogram.

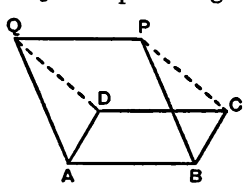


Fig. 357

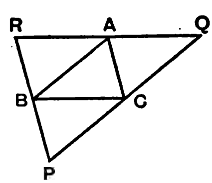


Fig. 358

8. In fig. 358, $\triangle$PQR is formed by drawing lines through A, B, C parallel to BC, CA, AB respectively. Prove that A, B, C are the mid-points of the sides of $\triangle$PQR.

[9] Two equal circles, centres A, B, cut at C, D. Prove that ACBD is a rhombus.

10. ABCD is a rhombus. If the bisector of $\angle DAC$ cuts CD at P, prove that $\angle DPA = 3\angle DAP$.

[11] ABCD is a parallelogram such that the bisectors of $\angle A$ and $\angle B$ meet on CD. Prove that $AB = 2BC$.

12. In $\triangle ABC$, $\angle A$ is a right angle; ABPQ and ACXY are squares lying outside $\triangle ABC$. Prove that PAX is a straight line.

13. In fig. 359, ABCD and APQR are squares. Prove that $BP = DR$.

[14] ABCD is a parallelogram; ABLM and BCHK are squares outside the parallelogram. Prove that (i) $\angle LBK = \angle BCD$, (ii) $KL = BD$.

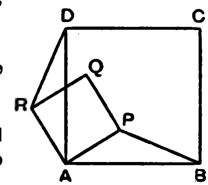


FIG. 359

[15] The side AB of the parallelogram ABCD is produced to X, and the bisector of $\angle CBX$ meets DA produced and DC produced at E and F. Prove that $DE = DF = BA + BC$.

16. In fig. 360, H, K are the mid-points of AB, AC, and HKP is a straight line. Prove that (i) $CP = AH$, (ii) CPHB is a parallelogram, (iii) $HK = \frac{1}{2}BC$.

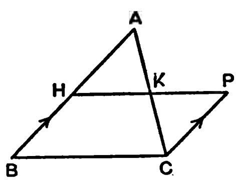


FIG. 360

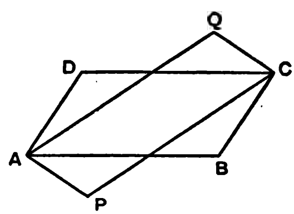


FIG. 361

17. In fig. 361, ABCD and APCQ are parallelograms. Prove that (i) AC, BD, PQ are concurrent, *i.e.* pass through the same point; (ii) PB is parallel to DQ.

[18] ABCD is a rhombus; PABQ is a straight line such that $PA = AB = BQ$. Prove that PD and QC when produced cut at right angles.

19. In fig. 362, Q is any point on the diagonal AC of a parallelogram ABCD. Prove that the areas of XQRD and PQYB are equal. [The area of a parallelogram is bisected by a diagonal.]

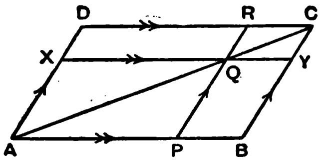



FIG. 362

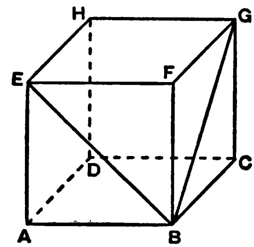



FIG. 363

20. If fig. 363 represents a cube, prove that $\angle$ EBG $=60°$.

21. If fig. 363 represents a cuboid, prove that $\angle$ EBG $=\angle$ DGB.

*22. N is the mid-point of the side AB of the rectangle ABCD; NP is drawn perpendicular to the plane of ABCD; P is joined to A, B, C, D. Point out, with reasons, pairs of congruent triangles.

*23. Fig. 364 represents a framework of 5 rods AB, BC, CD, BK, CK, jointed where they meet. The rods AB, DC can turn round the ends A, D which are fixed and the framework remains in one plane. P is a mark on the rod BC. Prove that P moves along the circumference of a circle whose centre lies on AD and find the path traced out by K. [Take E so that AE $=4''$, DE $=7''$.]

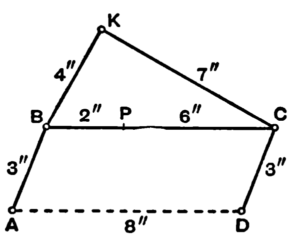



FIG. 364

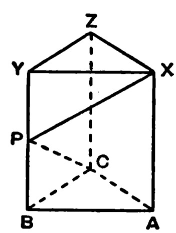



FIG. 365

*24. Fig. 365 represents a right prism whose ends ABC, XYZ are equilateral triangles; P is the mid-point of the edge BY. Prove that PC $=$ PX.

*25. The height of a right pyramid on a square base is equal to half the diagonal of the base. Prove that the faces are equilateral triangles.

*26. ABCD is a parallelogram; P is any point not in its plane; Q is the mid-point of PB; AQ is produced to R so that AQ = QR. Prove that RD bisects PC.

*27. ABCD is a square; E is a point on AC such that AE = AB; the line through E perpendicular to AC cuts BC, DC at F, G. Prove that $\angle FAG = 45°$.

*28. In $\triangle ABC$, $\angle C$ is a right angle; ABPQ is a square outside $\triangle ABC$; PN is the perpendicular from P to AC. Prove that PN = CA + CB.

*29. In fig. 366, ABCD is a square, $\angle APB$ is a right angle, and CXQ, DQ are drawn parallel to AP, BPX respectively. Prove that (i) PX = AP − PB, (ii) $\angle APQ = 45°$.

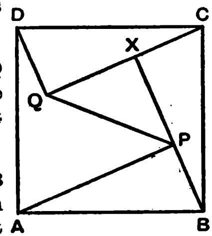


FIG. 366

*30. Three lines AOB, COD, EOF, not in the same plane, meet at O which is the mid-point of each. The parallelograms AECP, BFDQ are completed. Prove that (i) PQ passes through O, (ii) PF = EQ.

*31. BDG is the base of a tetrahedron and E is its vertex. Through each edge of the tetrahedron a plane is drawn parallel to the opposite edge, thus forming the parallelepiped in fig. 363. Use this construction to prove that the lines joining the mid-points of the opposite edges of a tetrahedron are concurrent and bisect each other.

What can you say about the opposite edges of the tetrahedron if the edges of the parallelepiped are all equal?

**Construction of Triangles and Quadrilaterals.** Examples of the construction of triangles and quadrilaterals from *simple* sufficient data have been given in Stage A. The sole purpose of Exercise 39 is to give material for the revision of this work; it may therefore be omitted if such revision is unnecessary.

# REVISION EXERCISE

## EXERCISE 39

**[Always make a neat sketch of the required figure and mark the data on it before you start to construct the figure.]**

Construct *when possible* $\triangle$ ABC from the given measurements in Nos. 1–9, *choosing your own unit of length.* If there are two different solutions, construct both. *If there is no solution, say so.*

1. (i) $a=3$, $b=4$, $c=5$; measure A.
   (ii) $a=3$, $b=4$, $c=8$; measure A.

[2] $a=5$, B $=30°$, C $=45°$; measure $b$.

3. $a=4$, A $=48°$, B $=33°$; measure $b$.

[4] (i) $a=7$, A $=110°$, B $=40°$; measure $b$.
   (ii) $a=5$, B $=125°$, C $=70°$; measure $b$.

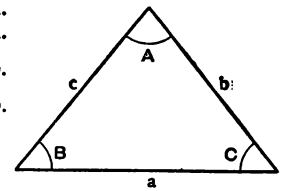


FIG. 367

[5] $b=7{\cdot}3$, $c=5{\cdot}4$, A $=125°$; measure $a$.

6. (i) $b=5$, $c=7$, C $=72°$; measure $a$.
   (ii) $b=6$, $c=4$, C $=40°$; measure $a$.
   (iii) $b=8$, $c=6$, C $=65°$; measure $a$.

[7] $a=b=6{\cdot}9$, A $=50°$; measure $c$.

8. $a=11{\cdot}2$, $b=7{\cdot}4$, A $=90°$; measure $c$.

9. $a:b:c=4:2:3$; measure A.

Construct the quadrilateral ABCD from the given measurements in Nos. 10–15, *choosing your own unit of length.*

[10] AB $=4$, BC $=4{\cdot}5$, CD $=3$, $\angle$B $=80°$, $\angle$C $=110°$; measure AD.

11. AB $=5$, AC $=6$, AD $=4$, BD $=7$, CD $=3$; measure BC.

[12] AB $=5$, BC $=6$, CD $=3$, DA $=4{\cdot}5$, $\angle$D $=100°$; measure $\angle$B.

13. $\angle$B $=70°$, $\angle$C $=95°$, $\angle$D $=105°$, AB $=5$, AD $=4$; measure BC.

[14] AB $=5$, $\angle$CAB $=35°$, $\angle$ABD $=47°$, $\angle$ACB $=65°$, $\angle$ADB $=54°$; measure CD.

15. AB $=$ BC $=3$, AD $=$ DC $=5$, $\angle$B $=120°$; measure $\angle$D.

[16] Construct an isosceles triangle with a base of 6 cm. and a vertical angle of 70°; measure its sides.

17. Construct an isosceles triangle with a base of 4·6 cm. and a height of 5 cm.; measure its vertical angle.

18. Draw a circle of radius 5 cm.; construct a triangle ABC such that A, B, C lie on the circumference, and $AB = 8$ cm., $AC = 7$ cm.; measure $\angle BAC$. [Two answers.]

**Construction of Triangles and Parallelograms.** For the construction of parallelograms, rectangles, etc., from sufficient data, familiarity with the properties discussed in pp. 148–155 is essential. These may be summarised as follows:—

(1) **In any parallelogram,**
- (*a*) opposite angles are equal;
- (*b*) opposite sides are equal;
- (*c*) the diagonals bisect each other;
- (*d*) each diagonal bisects the area.

(2) **A quadrilateral is a parallelogram, if**
- (*a*) both pairs of opposite angles are equal;
- (*b*) both pairs of opposite sides are equal;
- (*c*) one pair of opposite sides are equal and parallel;
- (*d*) the diagonals bisect each other.

(3) **In any rectangle,**
- (*a*) all the angles are right angles;
- (*b*) the diagonals are equal.

(4) **In any square,**
- (*a*) the diagonals are equal;
- (*b*) the diagonals cut at right angles;
- (*c*) the angle which each diagonal makes with each side of the square is 45°.

(5) **In any rhombus,**
- (*a*) all the sides are equal;
- (*b*) the diagonals cut at right angles;
- (*c*) the angles are bisected by the diagonals.

## CONSTRUCTION 7

**Describe a square on a given straight line.**

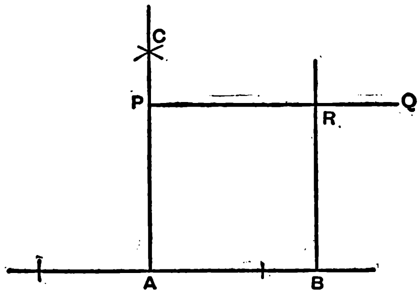



FIG. 368

**Given** a straight line AB.

**To construct** a square having AB as one side.

**Construction.** From A draw a line AC perpendicular to AB.
From AC cut off AP equal to AB.
Through P draw PQ parallel to AB.
Through B draw a line parallel to AP cutting PQ at R.
Then ABRP is the required square.

**Proof.** By construction, ABRP is a parallelogram.
Since $\angle$ BAP = 1 rt. $\angle$, ABRP is a rectangle.
Since AB = AP, the rectangle ABRP is a square.

**When constructing a figure from numerical data,**

(1) make a neat sketch of the required figure;
(2) mark on your sketch the given measurements;
(3) try to find or draw some *triangle* in the figure which can be constructed from the data, or by deductions from the data, see Examples 1–3.

### Examples for Oral Discussion

**1. Construct a parallelogram ABCD such that**

$$AB = 6 \text{ cm.}, \quad AC = 10 \text{ cm.}, \quad BD = 8 \text{ cm.}$$

**Measure AD.**

Draw a neat sketch of ABCD, showing the data on it, and let the diagonals cut at K.

(i) Mark on your sketch the lengths of AK and BK.
(ii) What part of the figure can now be constructed?
(iii) Complete the construction and measure AD.

2. **Construct a trapezium ABCD in which AB is parallel to DC, and**

AB = 8·5 cm., BC = 3·5 cm., CD = 4·5 cm., DA = 3 cm.

**Measure BD.**

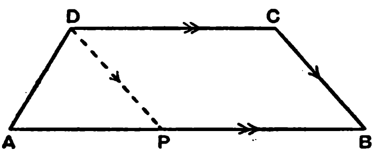



Fig. 369

Draw a neat sketch of ABCD, showing the data on it. In your sketch, draw DP parallel to CB to meet AB at P.

(i) Mark on your sketch the lengths of DP and AP.
(ii) What part of the figure can now be constructed?
(iii) Complete the construction and measure BD.

3. Construct a triangle ABC in which

$\angle B = 60°$, $\angle C = 40°$, perimeter = 9 cm.

Measure BC.

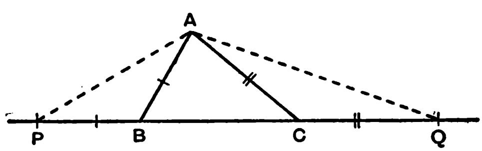



Fig. 370

Draw a neat sketch of $\triangle ABC$, showing on it the sizes of $\angle B$, $\angle C$, and complete fig. 370 in the way indicated by the markings, see p. 164.

In your sketch, produce CB to P so that BP = BA, and produce BC to Q so that CQ = CA. Join AP, AQ.

(i) Mark on your sketch the length of PQ and the sizes of $\angle$P, $\angle$Q.

(ii) What part of the figure can now be constructed?

(iii) B lies on the perpendicular bisector of AP. Why is this?

(iv) Complete the construction and measure BC.

## EXERCISE 40

[*In this exercise set-squares may be used for drawing parallels and perpendiculars.*]

1. Construct the rectangle ABCD, given that AB = 4 cm. and AC = 6 cm.; measure AD.

[2] Construct the parallelogram ABCD, given that AB = 4 cm., AD = 5 cm., AC = 6 cm.; measure $\angle$BAD.

3. Construct the rhombus ABCD, given that BD = 7 cm., $\angle$B = 40°; measure AC.

[4] Construct the square ABCD, given that AC = 5 cm.; measure AB.

5. Construct the rectangle ABCD, given that the diagonals intersect at an angle of 54° and that BD = 8 cm.; measure the sides.

[6] Construct a rhombus ABCD, given that AB = 5 cm., AC = 6 cm.; measure $\angle$BAD.

7. Construct a parallelogram ABCD, given that AB = 7 cm., AC = 10 cm., BD = 8 cm.; measure BC.

[8] Construct a parallelogram ABCD, given that AC = 4 in., BD = 5 in., and that the diagonals intersect at an angle of 50°; measure the longer side.

9. Construct the rhombus ABCD, given that AC = 6 cm., BD = 9 cm.; measure AB.

[10] Construct $\triangle$ABC, given that $\angle$A = 70°, $\angle$C = 35°, and the length of the perpendicular from A to BC is 2 in.; measure BC.

11. Construct $\triangle$ABC, given that $\angle$C = 68°, AB = 6 cm., and the length of the perpendicular from A to BC is 4 cm.; measure BC.

In Nos. 12–16, construct $\triangle$ ABC, choosing your own unit of length:

[12] $a+b=11$, $b+c=16$, $c+a=13$; measure $\angle$ A.

[13] A − B $=25°$, C $=55°$, $c=7$; measure $a$.

14. $b=c$, $a=4$, B − A $=24°$; measure $b$.

[15] A + B $=118°$, B + C $=96°$, $a=7$; measure $c$.

16. A : B : C = 1 : 2 : 3, $a=3$; measure $c$.

17. D is a point on the side BC of an equilateral triangle ABC. Given that BD = 3 cm. and $\angle$DAC = 40°, construct $\triangle$ ABC and measure BC.

18. Draw two parallel lines AB, CD, 5 cm. apart, and take any point O between AB and CD. Construct a straight line POQ cutting AB, CD at P, Q so that PQ = 6 cm.

In Nos. 19–24, construct a trapezium ABCD in which AB is parallel to DC.

19. AB = 8 cm., BC = 4 cm., CD = 3 cm., DA = 3·5 cm.; measure $\angle$ A.

[20] AB = 5 cm., BC = 6 cm., CD = 2 cm., DA = 4 cm.; measure $\angle$ A.

21. AB = 8 cm., CD = 5 cm., $\angle$ A = 72°, $\angle$ B = 40°; measure BC.

[22] AB = 4 cm., CD = 7 cm., $\angle$ A = 130°, $\angle$ B = 70°; measure AD.

23. AB = 6·5 cm., CD = 3 cm., AC = 7 cm., BD = 5 cm. Describe shortly your method. [In your sketch, complete the parallelogram CDBP.]

[24] AB = 4·5 cm., CD = 2·5 cm., AC = 4 cm., BD = 5 cm. Describe shortly your method.

25. The perpendicular distances between the opposite sides of a parallelogram are 3 cm., 4 cm., and one angle is 70°. Construct the parallelogram and measure one of its longer sides.

*26. The perpendicular distance between one pair of opposite sides of a parallelogram is 4 cm. and the lengths of the diagonals are 5 cm. and 8 cm. Construct the parallelogram. Describe shortly your method.

In Nos. 27–33, construct the triangle **ABC**, choosing your own unit of length.

***27.** $\angle A = 65°$, $\angle B = 70°$, $a + b + c = 12$; measure $a$.

***28.** $\angle A = 90°$, $a = 10$, $b + c = 13$; measure $b$. [In your sketch produce CA to P so that AP = AB; join BP; what is $\angle$CPB?]

***29.** $\angle B = 80°$, $b = 10$, $a + c = 13$; measure $c$.

***30.** $\angle A = 70°$, $c = 7$, $a + b = 14$; measure $a$. [Produce AC to P so that CP = CB. C is on the perpendicular bisector of BP; why?]

***31.** $\angle B = 35°$, $a = 8$, $b + c = 10$; measure $b$.

***32.** $\angle B = 25°$, $a = 9$, $c - b = 4$; measure $c$. [From AB cut off AP equal to AC; join CP.]

***33.** $\angle A = 70°$, $a = 9$, $b - c = 2$; measure $b$. [From AC cut off AP equal to AB; join BP. Calculate $\angle$CPB.]

***34.** Construct an isosceles triangle of height 5 cm. and perimeter 18 cm.; measure its base.

***35.** Construct a triangle ABC, given that BC = 1·7 in., the length of the perpendicular from A to BC is 1·2 in. and that the length of the line joining B to the mid-point of AC is 0·9 in. Describe shortly your method.

***36.** Construct a convex quadrilateral PQRS given that PR = 8 cm., RS = 6 cm., $\angle$QPR = 65° and that S is 3·5 cm. from PQ and that the diagonals cut at right angles. Describe shortly your method.

## Inequalities

The fundamental theorem on inequalities is as follows:—

**An exterior angle of a triangle is greater than either of the interior opposite angles.**

This can be regarded as a corollary of Theorem 8, p. 102, but can be proved without the use of parallels, see Appendix, p. 543; this proof provides an instructive rider on congruent triangles.

The symbol $>$ means *is greater than.*
The symbol $<$ means *is less than.*

Thus 5 in. $>$ 4 in. and $70° < 90°$.

## THEOREM 21

**If two sides of a triangle are unequal, the greater side has the greater angle opposite to it.**

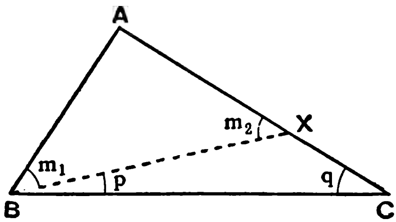



FIG. 371

**Given** a triangle ABC in which AC > AB.

**To prove that** $\angle$ABC > $\angle$ACB.

**Construction and Proof.** Since AC > AB, there is a point X on AC between A and C, such that AX = AB. Join BX.

With the notation in the figure,

since AB = AX *constr.*,
$m_1 = m_2$ *base* $\angle$*s, isos.* $\triangle$.

But $m_2 = p + q$, *ext.* $\angle$ *of* $\triangle$,

$\therefore$ $m_2 > q$,

$\therefore$ $m_1 > q$.

But $\angle$ABC > $m_1$ because X lies *between* A and C,

$\therefore$ $\angle$ABC > $q$, *i.e.* $\angle$ACB.

*Notice the combined heading,* "construction and proof." This is due to the fact that the data must be used to prove that the construction is possible.

### THEOREM 22 (Proof by Exhaustion)

**If two angles of a triangle are unequal, the greater angle has the greater side opposite to it.**

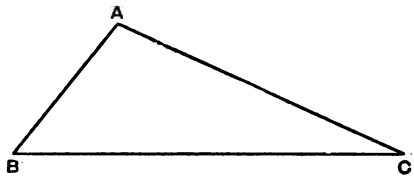


FIG. 372

**Given** a triangle ABC in which $\angle B > \angle C$.

**To prove that** $AC > AB$.

**Proof.** *Either* $AC < AB$

*or* $AC = AB$

*or* $AC > AB$.

If $AC < AB$, the angle opposite the greater side AB is greater than the angle opposite the smaller side AC,

$\therefore \angle C > \angle B$, contrary to what is given;

$\therefore$ AC is not less than AB.

If $AC = AB$, $\angle C = \angle B$, *base* $\angle s$, *isos.* $\triangle$,

which is contrary to what is given;

$\therefore$ AC is not equal to AB.

$\therefore$ AC is neither less than AB nor equal to AB,

$\therefore AC > AB$.

**Corollary.** In an obtuse-angled triangle, the greatest side is opposite to the obtuse angle; in a right-angled triangle, the hypotenuse is the greatest side.

This method of proof is called a **proof by exhaustion.**

It consists in taking in turn each possible supposition and proving that all *except one of them* are untrue. It then follows that the remaining supposition must be true. An alternative *direct* proof follows.

## THEOREM 22 (Direct Proof)

**If two angles of a triangle are unequal, the greater angle has the greater side opposite to it.**

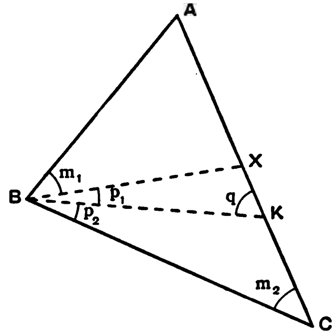



FIG. 373

**Given** a triangle ABC in which $\angle B > \angle C$.

**To prove that** $AC > AB$.

**Construction and Proof.** Since $\angle B > \angle C$, there is a point X on AC between A and C such that

$$\angle ABX = \angle C.$$

Let the bisector of $\angle XBC$ cut XC at K.
With the notation in the figure,

$$\angle ABK = m_1 + p_1,$$
$$\angle AKB = m_2 + p_2 \qquad \textit{ext. } \angle \textit{ of } \triangle.$$

But $m_1 = m_2$ and $p_1 = p_2$ *constr.*,

$$\therefore \angle ABK = \angle AKB;$$

but these are angles of $\triangle AKB$,

$$\therefore AK = AB.$$

But K lies between C and X and therefore between C and A,

$$\therefore AC > AK,$$
$$\therefore AC > AB.$$

## THEOREM 23

**Of all straight lines which can be drawn to a given straight line from a given point outside it, the perpendicular is the shortest.**

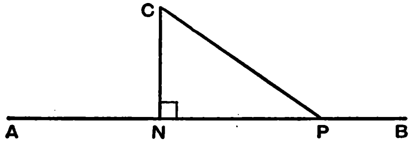



FIG. 374

**Given** a straight line AB, a point C outside AB, the perpendicular CN from C to AB, and any other point P on AB.

**To prove that** CN < CP.

**Proof.** In △CNP,

$$\angle N = 1 \text{ rt. } \angle \qquad \textit{given,}$$
$$\therefore \angle P + \angle C = 1 \text{ rt. } \angle \qquad \angle \textit{ sum of } \triangle = 2 \textit{ rt. } \angle s,$$
$$\therefore \angle P < 1 \text{ rt. } \angle,$$
$$\therefore \angle P < \angle N.$$

∴ in △CNP, the side opposite ∠P is less than the side opposite ∠N,

$$\therefore CN < CP.$$

**Corollary.** Conversely, if CK is the shortest straight line from C to any point of AB, then CK is perpendicular to AB.

For if CK is not perpendicular to AB, it is greater than the perpendicular from C to AB.

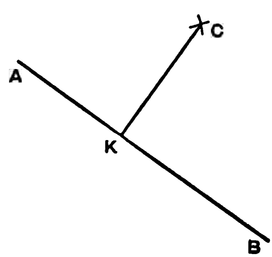



FIG. 375

## Distance of a Point from a Straight Line

*From a given point* P *outside a given straight line* AB, *one,* **and only one,** *straight line can be drawn perpendicular to* AB, *produced if necessary.*

A construction has been given showing how one perpendicular can be drawn, and if two perpendiculars could be drawn they would form with the given line a triangle in which two of the angles are right angles; this is impossible, because the sum of the three angles of any triangle is two right angles (see p. 102).

**Definition.** The length of the perpendicular PX from any point P to a straight line AB is called the **distance of P from AB.** It is the *shortest* distance of P from any point on AB.

A point P is said to be **equidistant** from two straight lines AB and CD if the perpendiculars PX, PY from P to AB, CD are equal.

NOTE. In fig. 376, the perpendicular from P to CD meets CD at a point on DC produced.

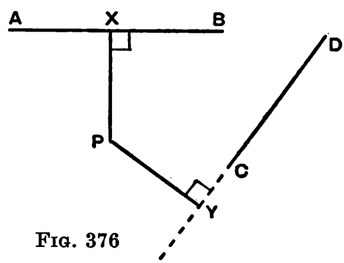


FIG. 376

**Distance between Two Points.** The distance between two points A, B is the length of the *straight* line AB, and it may reasonably be taken as true that this is the *shortest* distance from A to B because in fact it is as obvious as other assumptions that have previously been made (explicitly or implicitly). A "proof" is, however, added here because

(i) the construction and method of proof are useful in rider-work and in other constructions;

(ii) it is still set as a theorem in some examinations.

### THEOREM 24

**Any two sides of a triangle are together greater than the third side.**

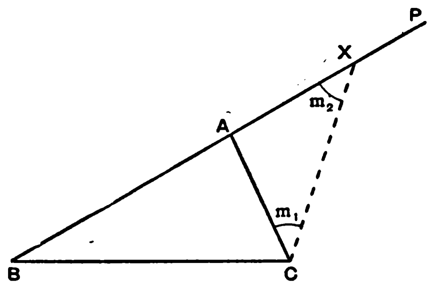


FIG. 377

**Given** a triangle ABC.

**To prove that** $BA + AC > BC$.

**Construction.** Produce BA to P and cut off AX from AP equal to AC.
Join CX.

**Proof.** With the notation in the figure,

since $AC = AX$ *constr.*,

$\therefore$ $m_1 = m_2$ *base* $\angle$*s, isos.* $\triangle$.

But $\angle BCX > m_1$,

$\therefore$ $\angle BCX > m_2$;

$\therefore$ in $\triangle BCX$, the side opposite $\angle BCX$ is greater than the side opposite $m_2$,

$\therefore$ $BX > BC$.

But $BX = BA + AX$

$= BA + AC$ *constr.*,

$\therefore$ $BA + AC > BC$.

**Corollary.** The *difference* between any two sides of a triangle is *less than* the third side.

## NUMERICAL EXAMPLES

### EXERCISE 41

1. In $\triangle$ ABC, AB = AC and $\angle$B = 62°. Which is the longer, AB or BC?

[2] The sides CA, CB of $\triangle$ABC are produced to H, K; the bisectors of $\angle$ABC, $\angle$ACB meet at I; $\angle$BAH = 126°, $\angle$ABK = 118°. Which is the longer, IB or IC?

3. In fig. 378, ABCD is a straight line; BQ, CQ are the bisectors of $\angle$PBD, $\angle$PCD. Which is the longer, (i) PB or PC, (ii) BC or QC?

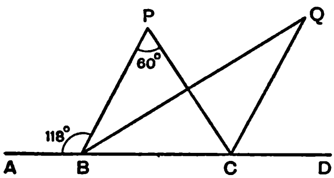



FIG. 378

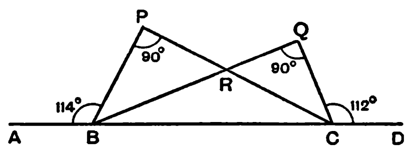



FIG. 379

4. In fig. 379, ABCD is a straight line. Which is the longer, (i) RB or RC, (ii) PB or PR?

[5] In $\triangle$ABC, $\angle$A = 60°, $\angle$B = 58°; AB, AC are produced to H, K; PB, PC are the bisectors of $\angle$HBC, $\angle$KCB. Which is the longer, (i) AB or AC, (ii) PB or PC?

[6] In $\triangle$ABC, $\angle$B = 30°, $\angle$C = 56°; the bisector of $\angle$BAC cuts BC at X. Arrange in order of length, the shortest first, AX, BX, CX.

[7] P is a point between B and C on the side BC of $\triangle$ABC, such that PA = PC; $\angle$C = 58°, $\angle$PAB = 33°. Which is the longer, PB or PC?

8. P is a point between B and C on the side BC of the equilateral triangle ABC. Arrange in order of length, the shortest first, the sides of $\triangle$ABP.

9. Is it possible to draw a triangle whose sides are of lengths (i) 2·5 cm., 3·5 cm., 6·5 cm.; (ii) 2 in., 3 in., 4 in.; (iii) 1 in., 2 in., 3 in.?

[10] How many unequal triangles can be drawn such that the lengths of two sides are 4 ft., 7 ft., and such that the length of the third side is a whole number of feet?

11. ABCD is a convex quadrilateral in which AB = 7 cm., BC = 2 cm., CD = 3 cm., DA = 4 cm. (i) Between what limits must the length of AC lie? (ii) Prove that $\angle DCB > \angle DBC$ and that $\angle ADB > \angle DAB$.

12. ABCD is a trapezium in which AB, DC are the parallel sides; AC cuts BD at K. If $\angle CAB = 41°$ and $\angle AKB = 100°$, find which is the greater, AC or BD.

*13. In fig. 380, RQ is parallel to BC. Prove that

(i) AR > PR > QC;
(ii) BP > PQ.

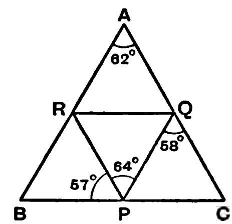


FIG. 380

*14. In $\triangle ABC$, $\angle B = 90°$, $\angle C = 29°$; prove that $AB < \frac{1}{2}AC$.

## EXERCISE 42

1. In fig. 381, AB = AC and BAP is a straight line. Prove that PB > PC.

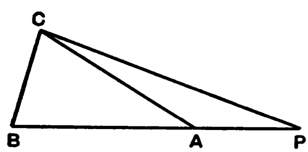


FIG. 381

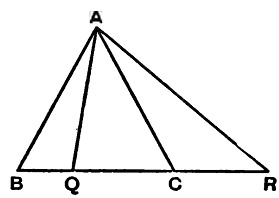


FIG. 382

2. In fig. 382, AB = AC and BQCR is a straight line. Prove that AQ < AC < AR.

[3] In $\triangle ABC$, BC > CA > AB. Prove that $\angle A > 60° > \angle C$.

4. In $\triangle ABC$, $\angle B = 90°$, $\angle C = 45°$. Prove that $BC > \frac{1}{2}AC$.

5. In $\triangle ABC$, the bisector of $\angle BAC$ cuts BC at D. Prove that BA > BD.

[6] In $\triangle ABC$, AB > AC and the bisector of $\angle BAC$ cuts BC at D. Prove that $\angle BDA$ is obtuse.

7. In $\triangle ABC$, AB > AC. If the bisectors of $\angle ABC$, $\angle ACB$ meet at I, prove that IB > IC.

[8] ABCD is a convex quadrilateral in which $\angle B = \angle D$ and $AB > AD$. Prove that $CB < CD$. [Join BD.]

9. In fig. 383, $AB > AD$ and $\angle B = \angle D$. Which is the greater, CB or CD? Give reasons.

10. ABCD is a quadrilateral. Prove that $AB + BC + CD > AD$. [Join AC.]

[11] Prove that any side of a triangle is less than half the perimeter of the triangle.

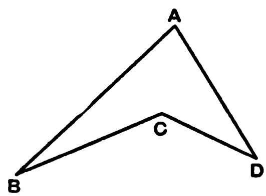



FIG. 383

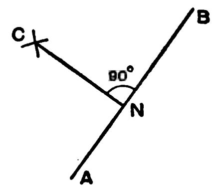



FIG. 384

12. In fig. 384, prove that, if a circle is drawn with C as centre and CN as radius, it does not meet AB at any point except N.

13. O is any point inside the triangle ABC. Prove that $\angle BOC > \angle BAC$. [Produce BO to meet AC at K.]

[14] D is any point on the side BC of $\triangle ABC$. Prove that $AD < \frac{1}{2}(BC + CA + AB)$.

15. P is any point inside the convex quadrilateral ABCD. Prove that $PA + PB + PC + PD$ is not less than $AC + BD$. What is the position of P if $PA + PB + PC + PD = AC + BD$?

16. In the convex quadrilateral ABCD, AB is the greatest side and CD is the least side. Prove that $\angle D > \angle B$. [Join BD.]

17. In fig. 385, $\angle B$ is acute and $\angle B = 2\angle C$. Prove that $AC < 2AB$.

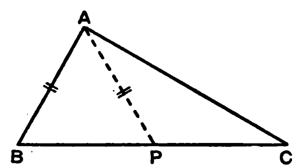



FIG. 385

[18] The sides AB, AC of $\triangle ABC$ are produced to P, Q; the bisectors of $\angle PBC$, $\angle QCB$ meet at X. If $AB > AC$, find which is the greater, XB or XC. Give reasons.

19. If in fig. 386, $EB = EC$, prove that $ED > EA$.

20. O is any point inside the triangle ABC. Prove that $BA + AC > BO + OC$. [Produce BO to meet AC at K.]

[21] ABCD is a trapezium in which AB, DC are parallel; AC cuts BD at K. If $AC > BD$, prove that $AK > KB$. [Draw CP parallel to DB to meet AB produced at P.]

22. D is the mid-point of the side BC of $\triangle ABC$. If $AB > AC$, prove that $\angle BAD < \angle DAC$. [Produce AD to P so that $AD = DP$. Join BP.]

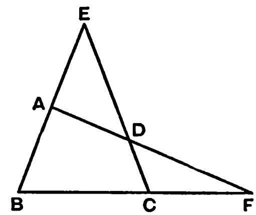


FIG. 386

[23] D is the mid-point of the side BC of $\triangle ABC$. Prove that $AD < \frac{1}{2}(AB + AC)$.

*24. In $\triangle ABC$, $AB > AC$. If the bisector of $\angle BAC$ cuts BC at D, prove that $BD > DC$. [Produce AC to P so that $AP = AB$. Join DP.]

*25. In fig. 387, A′ is the *image* of A in the straight line CD, *i.e.* CD bisects AA′ at right angles; A′EB is a straight line. Prove that (i) $\angle AEC = \angle BED$; (ii) $AP + PB > AE + EB$. [Join A′P.]

This is called the **light-path theorem.** If a ray of light from a source A is reflected in a mirror CD so as to travel to B, it follows the shortest path AEB, and for this path the angle of incidence, $\angle AEC$, is equal to the angle of reflection, $\angle BED$.

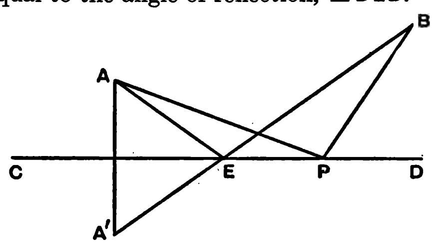


FIG. 387

*26. O is any point inside the triangle ABC. Prove that $OA + OB + OC > \frac{1}{2}(BC + CA + AB)$. [No construction.]

*27. Prove that the sum of the lengths of the diagonals of a convex quadrilateral is greater than the semi-perimeter.

## EQUAL INTERCEPT THEOREMS

**Definition.** If a transversal LM cuts two lines AB, CD at H, K, see fig. 388, then HK is called the **intercept** made by AB and CD on LM.

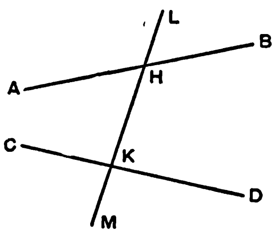



Fig. 388

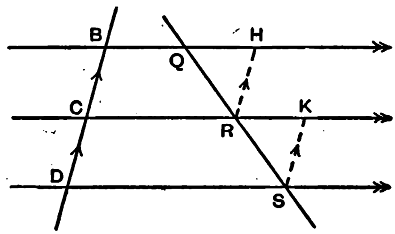



Fig. 389

### Examples for Oral Discussion

**1. In fig. 389, BC = CD and arrows indicate that lines are parallel.** Prove that QR = RS.

(i) Give the reasons why RH = SK.

(ii) Use the triangles RHQ, SKR to complete the proof.

**2.** (i) What figure is obtained from fig. 389 by making Q the same point as B? Draw it.

(ii) Complete the sentence: the straight line drawn through the mid-point of the side BD of △BDS, parallel to DS, . . .

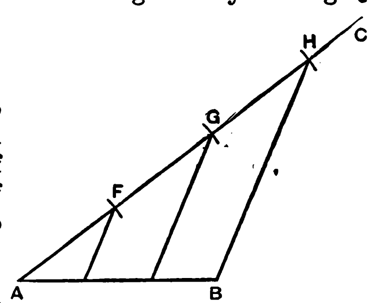



Fig. 390

**3.** Explain how to divide a given line AB into 3 equal parts without measuring it. Does fig. 390 suggest a way of doing so? Give reasons.

**4. In fig. 391, AH = HB and AK = KC. Prove that**

**(i) HK is parallel to BC, (ii) HK = $\frac{1}{2}$BC.**

Draw CP parallel to BA to meet HK produced at P.

(i) Explain why $\triangle$CKP$\equiv\triangle$AKH.

(ii) Prove that CP = BH. What then follows from the fact that CP is equal and parallel to BH?

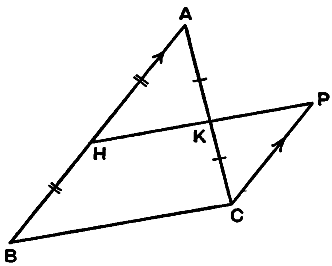



FIG. 391

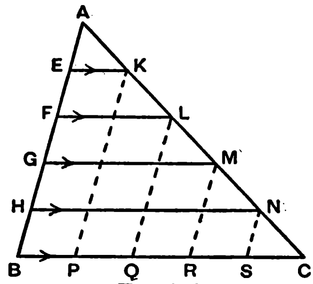



FIG. 392

5. In fig. 392, the side AB of $\triangle$ABC is divided into 5 equal parts by the lines EK, FL, GM, HN drawn parallel to BC to cut AC at K, L, M, N.

Prove that EK = $\frac{1}{5}$BC, FL = $\frac{2}{5}$BC,
GM = $\frac{3}{5}$BC, HN = $\frac{4}{5}$BC.

Draw KP, LQ, MR, NS parallel to AB to cut BC at P, Q, R, S.

(i) What can you say about the points K, L, M, N?

(ii) What can you say about the points P, Q, R, S?

6. If, in fig. 392, AH = $\frac{7}{10}$AB, what can you say about the length of HN?

## NUMERICAL EXAMPLES

### EXERCISE 43

1. In fig. 393, if AP = PB = 2 in., BC = 3 in., and AC = 3·6 in., find the lengths of AQ and PQ.

[2] X, Y, Z are the mid-points of the sides BC, CA, AB of $\triangle$ABC. If BC = 5 in., CA = 6 in., AB = 7 in., find the lengths of XY, YZ and ZX.

3. In fig. 393, if AP = 2 in., PB = 3 in., BC = 4 in., and AC = 4·5 in., find the lengths of AQ and PQ. [Look at fig. 392.]

[4] In fig. 393, if AP = $\frac{3}{7}$AB, what can you say about AQ and about PQ? Give reasons.

5. In fig. 394, find the values of $a$ and $b$.

[6] In fig. 394, find $x$ if $y = 12$.

7. In fig. 394, find $z$ if $y = 12$.

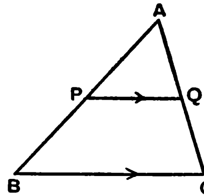


FIG. 393

[8] If, in fig. 393, QR is drawn parallel to AB to meet BC at R, and if BP = 10 cm., AQ = 12 cm., BR = 15 cm., and RC = 20 cm., find QC and AP.

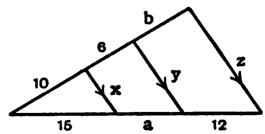


FIG. 394

9. In fig. 395, if AC = CB = 4″, find CR. [Find CK and KR.]

10. In fig. 395, if AC = 3″ and CB = 6″, find the length of CR.

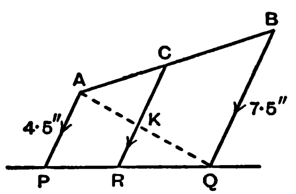


FIG. 395

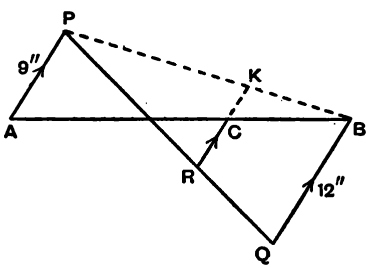


FIG. 396

11. In fig. 396, if AC = CB = 10″, find CR. [Find CK and KR.]

12. In fig. 396, if AC = 8″, CB = 4″, find the length of CR.

[13] The tops A, C of two vertical poles AB, CD which stand on level ground are joined by a rod. If AB = 5 ft. and CD = 12 ft., find the height of the mid-point of AC above the ground.

14. At the corners of a rectangular horizontal court, vertical poles of heights 12, 10, 6, 7 ft. are erected in order round the court. The tops are joined diagonally by straight wires. Will the wires intersect? If not, what alteration in the height of the shortest pole is necessary to make them do so?

## THEOREM 25

**The straight line joining the mid-points of two sides of a triangle is parallel to the third side and equal to one-half of it.**

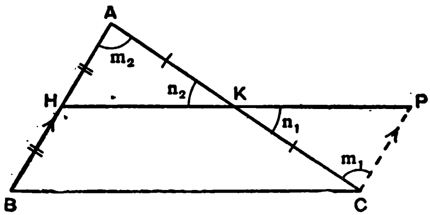



FIG. 397

**Given** the mid-points H, K of the sides AB, AC of $\triangle$ABC.

**To prove that** (i) HK $\parallel$ BC,

(ii) HK $= \frac{1}{2}$BC.

**Construction.** Through C draw CP parallel to BA to meet HK produced at P.

**Proof.** (i) With the notation in the figure,

in $\triangle$s CKP, AKH,

$m_1 = m_2$ *alt.* $\angle$*s*, CP $\parallel$ BA,

$n_1 = n_2$ *vert. opp.* $\angle$*s*,

CK = AK *given*,

$\therefore$ $\triangle$s $\begin{matrix}\text{CKP}\\ \text{AKH}\end{matrix}$ are congruent ASA,

$\therefore$ CP = AH and PK = HK,

but AH = BH *given*, $\therefore$ CP = BH.

Also CP is drawn parallel to BH,

$\therefore$ the lines CP, BH are equal and parallel,

$\therefore$ BCPH is a parallelogram,

$\therefore$ HP $\parallel$ BC, *i.e.* HK $\parallel$ BC.

(ii) Also BC = HP *opp. sides* $\parallel$*gram*,

but HK = KP *proved*, $\therefore$ HK $= \frac{1}{2}$HP,

$\therefore$ HK $= \frac{1}{2}$BC.

*Abbreviation for reference:* mid-point theorem.

## THEOREM 26

**The straight line drawn through the middle point of one side of a triangle parallel to another side bisects the third side.**

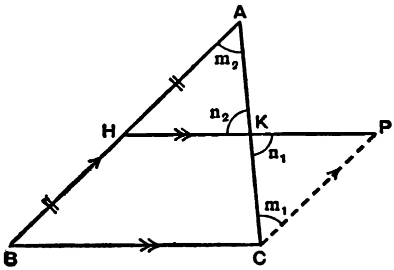



FIG. 398

**Given** the mid-point H of the side AB of $\triangle$ABC, and a line through H parallel to BC cutting AC at K.

**To prove that** AK = KC.

**Construction.** Through C draw CP parallel to BA to meet HK produced at P.

**Proof.** Since HP $\parallel$ BC *given*,
and CP $\parallel$ BH *constr.*,

BCPH is a parallelogram,

$\therefore$ CP = BH *opp. sides* $\parallel$gram,

but BH = HA *given*, $\therefore$ CP = HA.

In $\triangle$s CPK, AHK, with the notation in the figure,

$m_1 = m_2$ *alt.* $\angle$*s*, CP $\parallel$ BA,
$n_1 = n_2$ *vert. opp.* $\angle$*s*,
CP = AH *proved*,

$\therefore$ $\triangle$s $\begin{matrix}\text{CPK}\\ \text{AHK}\end{matrix}$ are congruent AAS,

$\therefore$ CK = AK.

*Abbreviation for reference:* Intercept theorem.

## THEOREM 27

**If three or more parallel straight lines make equal intercepts on a given transversal, they make equal intercepts on any other transversal.**

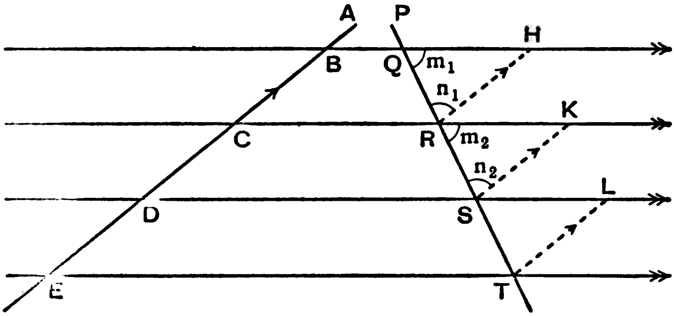



FIG. 399

Given the parallel lines BQ, CR, DS, ET, . . . cutting a transversal at B, C, D, E, . . . so that BC = CD = DE = . . . and cutting another transversal at Q, R, S, T, . . .

**To prove that** QR = RS = ST = . . .

**Construction.** Through R, S, T, . . . draw lines parallel to BCDE to meet BQ, CR, DS, . . . produced, if necessary, at H, K, L, . . .

Since BH ∥ CR *given*,
and RH ∥ CB *constr.*,
CRHB is a parallelogram,
∴ RH = CB *opp. sides* ∥*gram.*

Similarly it may be proved that

SK = DC,

but BC = CD *given*, ∴ RH = SK.

∴ in △s RHQ, SKR, with the notation in the figure,

$m_1 = m_2$ *corr.* ∠*s*, QH ∥ RK,
$n_1 = n_2$ *corr.* ∠*s*, RH ∥ SK,
RH = SK *proved*,

$\therefore$ $\Delta$s $\begin{matrix} RHQ \\ SKR \end{matrix}$ are congruent AAS,

$\therefore$ $QR = RS$.

Similarly it may be proved that $RS = ST$, etc.

*Abbreviation for reference:* Intercept theorem.

**Note.** Theorem 26 is a special case of Theorem 27, and could be regarded as a corollary of Theorem 27; separate proofs are given for examination purposes.

The construction used for Theorem 25 is the same as is used for Theorem 26 and Theorem 27; there are other constructions which could be used, but it is less confusing if the same construction is used for each of this group of theorems.

## Examples for Oral Discussion

**1. In fig. 400, $AC = CB$ and the parallel lines $AP$, $CR$, $BQ$ meet another transversal at $P$, $R$, $Q$. Prove that**

$$CR = \tfrac{1}{2}(AP + BQ).$$

Join $AQ$ and let it cut $CR$ at $K$.

(i) Explain why $AK = KQ$.

(ii) What do you know about the length of $CK$?

(iii) What do you know about the point $R$ and the length of $KR$?

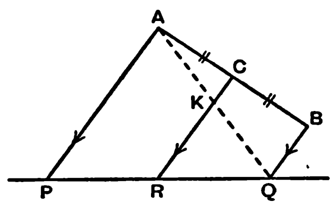


FIG. 400

**2. In fig. 401, $P$, $Q$, $R$, $S$ are the mid-points of the sides of any quadrilateral $ABCD$. Prove that $PQRS$ is a parallelogram.**

Join $AC$.

(i) Explain why $PQ$ is parallel to $AC$.

(ii) What do you know about the length of $PQ$?

(iii) What do you know about $SR$?

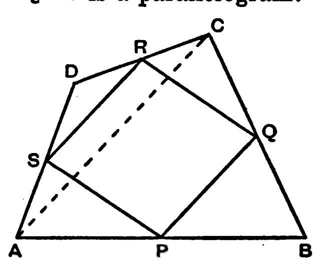


FIG. 401

3. Find out whether the statement in No. 2 applies to a quadrilateral ABCD which is not convex, see fig. 383, p. 175. Does it apply to a quadrilateral ABCD if ABC, ADC are in different planes?

**4. In fig. 402, F is the mid-point of the hypotenuse AB of the right-angled triangle ABC. Prove that CF $= \frac{1}{2}$AB.**

Draw FN parallel to BC to meet AC at N.

(i) What do you know about the length of AN?

(ii) Explain why $\triangle$FNC $\equiv$ $\triangle$FNA.

Complete the proof.

5. What can you say about the circle whose diameter is the hypotenuse of a right-angled triangle?

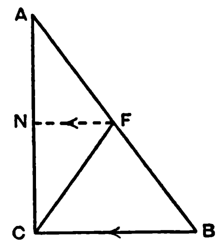


FIG. 402

## EXERCISE 44

1. In fig. 403, P, Q, R are the mid-points of BC, CA, AB; prove that PQAR is a parallelogram.

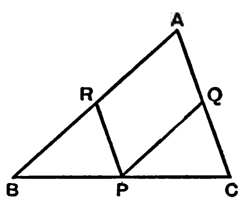


FIG. 403

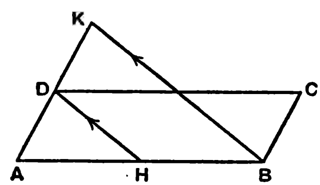


FIG. 404

2. In fig. 404, ABCD is a parallelogram and H is the mid-point of AB. Prove that DK = BC.

3. P, Q are the mid-points of the sides AB, DC respectively of the parallelogram ABCD; PD, BQ cut AC at H, K respectively. (i) Explain why PBQD is a parallelogram. (ii) Prove that AH = HK = KC.

[4] In fig. 393, p. 179, if AP = 2PB, prove that AQ = 2QC. [Draw another parallel through the mid-point of AP.]

5. In fig. 405, H, K, X, Y are the mid-points of PB, PC, QB, QC. Prove that $HK = XY$.

6. In fig. 406, $AD = DB$. Prove that $AH = BK$.

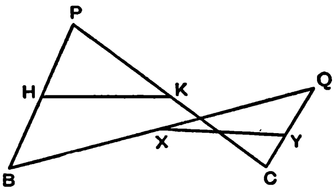



FIG. 405

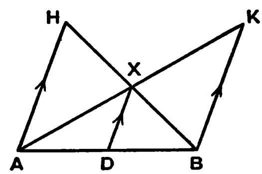



FIG. 406

[7] Four parallel lines cut one transversal at A, B, C, D and cut another transversal at P, Q, R, S. If $AB = CD$, prove that $PQ = RS$. [Through Q, S draw QH, SK parallel to DA to meet AP, CR, produced if necessary, at H, K.]

8. In fig. 407, $AH = HK = KB$; prove that (i) $HP = \frac{1}{3}BC$; (ii) $KQ = \frac{2}{3}BC$. [Draw PX and QY parallel to AB.]

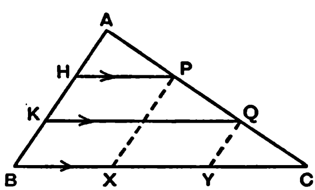



FIG. 407

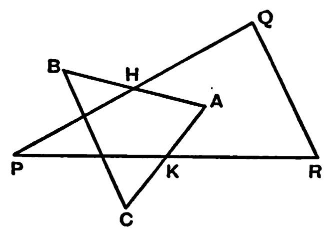



FIG. 408

9. In fig. 408, H is the mid-point of AB and of PQ; K is the mid-point of AC and of PR. Prove that $QR = BC$.

[10] P is the mid-point of the side BC of $\triangle ABC$; Q is the mid-point of AP; BQ produced meets AC at R. Prove that $AC = 3AR$. [Draw PK parallel to BR to meet AC at K.]

[11] In $\triangle ABC$, $\angle B = 1$ right angle; BCX is an equilateral triangle. Prove that the line from X parallel to AB bisects AC.
[There are two cases.]

12. In fig. 396, p. 179, if $AC = CB$, prove that for any lengths of AP and BQ, $CR = \frac{1}{2}(BQ \sim AP)$.

13. AB is a given straight line and O is a given point outside AB; P is a variable point on AB. Prove that the mid-point of OP lies on a fixed straight line.

14. A and B are given points; P is a variable point on a given circle, centre B. Prove that the mid-point of AP lies on a fixed circle whose centre is the mid-point of AB.

15. If the diagonals of a quadrilateral cut at right angles, prove that the mid-points of the four sides are the vertices of a rectangle.

[16] If the diagonals of a quadrilateral are equal, prove that the mid-points of the four sides are the vertices of a rhombus.

17. ABCD is a rectangle; V is any point not in its plane; L, M, N, P are the mid-points of VA, VB, VC, VD. Prove that L, M, N, P are coplanar and are the vertices of a rectangle.

18. Q, R are the mid-points of the sides AB, BC of $\triangle$ABC; the perpendiculars AD, BE from A, B to BC, AC intersect at H; P is the mid-point of AH. Prove that $\angle$PQR is a right angle.

19. In fig. 409, O is the centre of the circle; AOBD is a straight line such that AB = BD. Prove that PR = RD. [Join AP.]

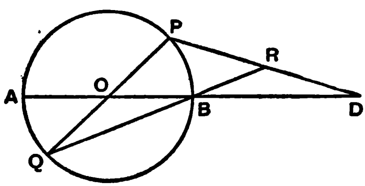



FIG. 409

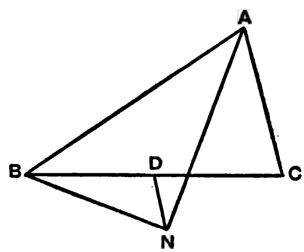



FIG. 410

20. In fig. 410, BN is the perpendicular from B to the bisector of $\angle$BAC; D is the mid-point of BC. Prove that DN $= \frac{1}{2}$(AB − AC). [Produce BN and AC to cut at K.]

[21] D is the mid-point of the side BC of $\triangle$ABC; CA is produced to E. If BR is the perpendicular from B to the bisector of $\angle$BAE, prove that DR $= \frac{1}{2}$(AB + AC).

22. D is the mid-point of the side BC of $\triangle$ABC; BR, CS are the perpendiculars from B, C to any straight line passing through A. Prove that DR = DS. First take the case when the line through A lies outside the angle BAC and then the case when it lies inside $\angle$BAC. [Draw the perpendicular DK from D to RAS.]

[23] Q, R are the mid-points of the sides AC, AB of $\triangle$ ABC; AD is the perpendicular from A to BC. Prove that $\angle$ RDQ = $\angle$ BAC. [Look at Example 4, p. 184.]

24. ABCD is a parallelogram and HK is a straight line outside the parallelogram. AP, BQ, CR, DS are the perpendiculars from A, B, C, D to HK. Prove that AP + CR = BQ + DS [Let AC cut BD at O; draw ON perpendicular to HK. Look at Example 1, p. 183.]

*25. BP, CQ are the bisectors of $\angle$ B, $\angle$ C of $\triangle$ ABC; AH, AK are the perpendiculars from A to BP, CQ; prove that KH is parallel to BC. [Produce AH, AK to cut BC at X, Y.]

*26. A square box ABCD rests in the rack of a railway carriage with one edge against the wall, see fig. 411. The point of contact E is 11 in. from the wall; E is the mid-point of AB, and A is 5 in. from the wall. Find the distances of B and C from the wall.

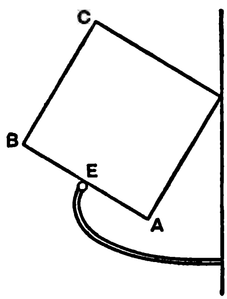



FIG. 411

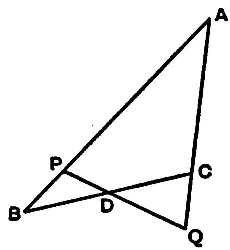



FIG. 412

*27. In fig. 412, D is the mid-point of BC; PDQ is a straight line such that AP = AQ. Prove that AP = $\frac{1}{2}$(AB + AC).

[*Hint*: D is the mid-point of BC; therefore with BC as one side, make a triangle for which the mid-point theorem can be used.]

*28. The diagonals AC, BD of the square ABCD intersect at K; the bisector of $\angle$ BAC cuts BK at X and cuts BC at Y. Prove that CY = 2KX. [Draw KP parallel to CB to cut AX at P.]

*29. In a tetrahedron ABCD, the plane angles at each of the corners A, B, C add up to 2 right angles. Prove that its opposite edges are equal. [Draw the net of the tetrahedron.]

*30. Prove that the lines joining the mid-points of opposite edges of a tetrahedron meet at a point and bisect each other.

## CONSTRUCTION 8

**Divide a given straight line into any given number of equal parts.**

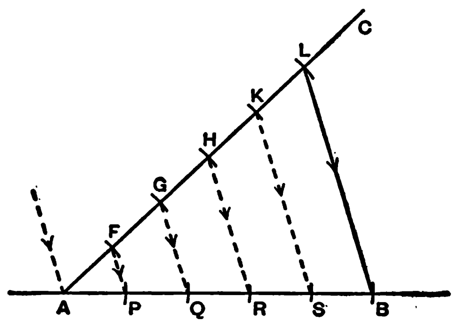



FIG. 413

**Given** a line AB.

**To construct** points dividing AB into any given number of equal parts, say 5.

**Construction.** From A draw any line AC making any convenient angle with AB, and from AC cut off equal lengths AF, FG, GH, . . . the number of such lengths being the required number of equal parts, in this case 5. Let the equal lengths be AF, FG, GH, HK, KL. Join LB, and through F, G, H, K draw lines parallel to LB to meet AB at P, Q, R, S. Then AP, PQ, QR, RS, SB are the required equal parts.

**Proof.** Since the parallel lines FP, GQ, HR, KS, LB, together with a parallel through A, make equal intercepts on AC, they also make equal intercepts on AB.

$$\therefore\ AP = PQ = QR = RS = SB.$$

## Examples for Oral Discussion

**Plain Scales.** The meaning of a scale has been explained on p. 64.

1. Draw a scale of $\frac{1}{10}$ to read feet and inches and long enough to measure 4 ft.

1 ft. is represented by $\frac{1}{10}$ ft.

$\therefore$ 4 ft. is represented by 0·4 ft., $=4{\cdot}8$ in.

Draw a line AB, 4·8 in. long, and use a construction to divide it into 4 equal parts, each of which then represents 1 ft. Use a construction to divide the first of these parts AC into 12 equal parts, each of which then represents 1 in.

2. Draw a scale, 8 in. to 1 mile, to read to 10 yd. and long enough to measure 1000 yd.

*First Method.* 1760 yd. is represented by 8 in.

$$\therefore \text{ 1000 yd. is represented by } \frac{8 \times 1000}{1760} \text{ in.,} \simeq 4{\cdot}55 \text{ in.}$$

Draw a line AB, 4·55 in. long, and use a construction to divide it into 10 equal parts, each of which then represents 100 yd. Use a construction to divide the first of these parts AC into 10 equal parts, each of which then represents 10 yd.

*Second Method.* Draw a line AD, 8 in. long; this represents 1760 yd.

$$1000 : 1760 = 25 : 44.$$

Use a construction to find a point C on AD such that

$$AC : AD = 25 : 44.$$

Then AC represents 1000 yd. Proceed as before.

3. Draw a scale of $\frac{1}{16}$ to read feet and inches and long enough to measure 6 ft.

4. Draw a scale, $1\frac{1}{2}$ in. to 1 ft., to read feet and inches and long enough to measure 4 ft.

5. Draw a scale of 1 : 20,000 to read to 100 metres and long enough to measure 3 km.

**6.** Draw a scale, 1 ft. to 1 mile, to read to 10 yd. and long enough to measure 1000 yd.

**7.** Draw a scale of 1 : 2500 to read to half-chains and long enough to measure 2 furlongs.

**8. The Diagonal Scale.** With a ruler graduated in tenths of an inch, it is possible to measure a length in inches correct to one place of decimals and to make an estimate to two places of decimals. More accurate measurements can be made by using a diagonal scale, see fig. 414.

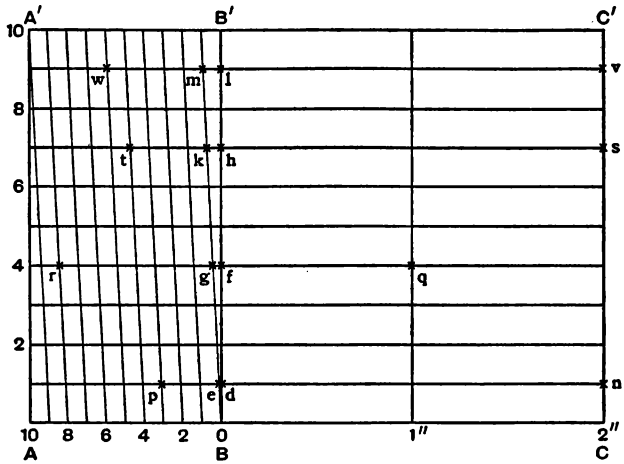



FIG. 414

ABC represents a line 3 in. long; the portion AB which represents 1 in. is divided into tenths of an inch. A′B′ is equal and parallel to AB and is divided into tenths of an inch, and AA′ is divided into 10 equal parts. Parallel lines are drawn as shown.

(i) Read off the lengths of *de*, *fg*, *hk*, *lm*.

(ii) Read off the lengths of *np*, *qr*, *st*, *vw*.

(iii) How can you use the diagonal scale to obtain the following lengths: 0·32 in., 1·56 in., 2·78 in.?

9. Construct a diagonal scale to show eighths and sixty-fourths of an inch.

10. Construct a diagonal scale, 1 in. to 1 mile, to read miles, furlongs and chains up to 4 miles.

**Definitions.** (1) Three or more straight lines are said to be **concurrent** if they pass through the same point.

(2) The straight line joining any vertex of a triangle to the mid-point of the opposite side is called a **median** of the triangle.

### Examples for Oral Discussion

Nos. 1–7 refer to fig. 415, in which E, F, G are the mid-points of AC, AB, AH. Copy this figure.

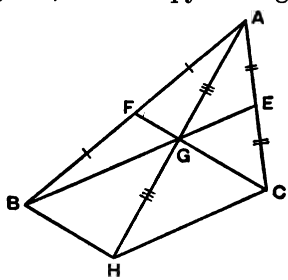



FIG. 415

1. What two facts do you know about EG?

2. Explain why BGCH is a parallelogram.

3. Let AGH cut BC at D, show this on your figure and explain why BD = DC.

This proves that **the three medians of any triangle are concurrent.** The point of intersection of the medians is called the **centroid** of the triangle.

4. Prove that $DG = \frac{1}{2}GA$ and that $DG = \frac{1}{3}DA$.

5. What are the corresponding facts about EG and FG?

6. If the lengths of the medians AD, BE, CF of $\triangle ABC$ are 3 cm., 4·5 cm., 6 cm., find the lengths of the sides of $\triangle GHC$, and then construct $\triangle ABC$.

7. Construct fig. 415, given that BE = 7·5 cm., CF = 6 cm., BC = 7 cm.

## THEOREM 28

**(1) The three medians of a triangle are concurrent.**

**(2) The point at which the medians intersect is one-third of the way along each median, measured towards the vertex.**

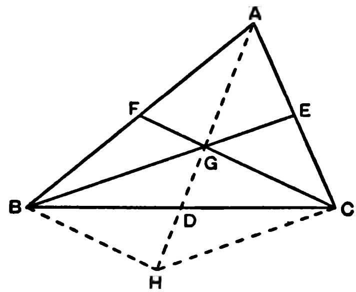



FIG. 416

**Given** the mid-points E, F of the sides AC, AB of $\triangle$ABC and that BE, CF cut at G.

**To prove that** if AG produced cuts BC at D,

(1) $BD = DC$.

(2) $DG = \frac{1}{3}DA$, $EG = \frac{1}{3}EB$, $FG = \frac{1}{3}FC$.

**Construction.** Produce AGD to H so that $AG = GH$. Join BH, CH.

**Proof.** (1) Since $AF = FB$ and $AG = GH$,

FG is parallel to BH *mid-point theorem.*

Since $AE = EC$ and $AG = GH$,

EG is parallel to CH *mid-point theorem.*

Since FGC $\parallel$ BH and EGB $\parallel$ CH,

GCHB is a parallelogram,

$\therefore$ the diagonals GH, BC bisect each other,

$\therefore$ $BD = DC$.

(2) For the same reason, $GD = DH$,

$\therefore$ $GH = 2GD$,

but $AG = GH$ *constr.*, $\therefore AG = 2GD$,
add to each $GD$, $\therefore AD = 3GD$, $\therefore DG = \frac{1}{3}DA$.

Similarly, it may be proved that

$$EG = \tfrac{1}{3}EB \quad \text{and} \quad FG = \tfrac{1}{3}FC.$$

*Abbreviation for reference:* centroid theorem.

## EXERCISE 45

1. If the medians AD, BE, CF of the triangle ABC meet at G, prove that G is the centroid of $\triangle$DEF.

2. ABCD is a parallelogram; P is the mid-point of AB; CP cuts BD at Q; prove AQ produced bisects BC. [Join AC.]

3. AD is a median of $\triangle$ABC; DA is produced to K so that $AK = 2DA$. Prove that BA produced bisects CK.

[4] ABCD is a parallelogram; BD is produced to P so that $BD = DP$. Prove CD produced bisects AP, and AD produced bisects CP.

5. If G is the centroid of $\triangle$ABC and if $AG = BC$, prove that $\angle$BGC is a right angle.

In Nos. 6–12, AD, BE, CF are the *medians* of $\triangle$ABC.

6. Construct $\triangle$ABC, given that $AB = 5$ cm., $AC = 4$ cm., $AD = 3{\cdot}5$ cm.; measure BC. [Can you draw ‖gram CABP?]

[7] Construct $\triangle$ABC, given that $BE = 6$ cm., $CF = 5{\cdot}4$ cm., $BC = 5$ cm.; measure AD.

8. Construct $\triangle$ABC, given that $AD = 6$ cm., $BE = 7{\cdot}5$ cm., $CF = 9$ cm.; measure BC.

9. Prove that $2BE + 2CF > 3BC$.

[10] Prove that $2AD + 3BC > 4BE$. [Consider $\triangle$BDG.]

11. Prove that $4(AD + BE + CF) > 3(BC + CA + AB)$.

[12] Prove that $BE + CF > AD$. [See fig. 416.]

*13. ABCD is a tetrahedron; P, Q are the centroids of the faces BCD, ACD. Prove that the straight lines AP, BQ intersect, and if G is their point of intersection, prove that (i) PQ is parallel to AB and equal to $\frac{1}{3}$AB, (ii) $PG = \frac{1}{4}PA$. [Join A and B to the mid-point of CD.]

*14. ABCD is a tetrahedron; P, Q, R, S are the centroids of the faces opposite A, B, C, D respectively. Prove that AP, BQ, CR, DS are concurrent.

## LOCI

If we look at the tip of the seconds-hand of a watch we see that it occupies a series of different positions in the course of each minute, and if we combine together all these different positions we obtain the circumference of a circle which the tip of the seconds-hand traces out each minute. This *aggregate* of all possible positions of the tip is called its **locus.** The reader is no doubt familiar with the word "aggregate" in connection with cricket scores; here we use it to mean the resulting curve obtained by combining all the different positions of a small object moving according to some given law.

When it is stated that the locus of a small object which moves about subject to some given law is a certain curve, two complementary ideas are involved:

(1) The position of every point on the curve satisfies the given law.

(2) Every position of the object which satisfies the given law lies on the curve.

It often happens that the conditions of the problem prevent the object from describing the whole of a curve; in this case it must be stated what part of the curve forms the locus.

Suppose, for example, a door is capable of being opened through an angle of 110°, but no more, what is the locus of a small mark on the handle?

The locus is an arc AB of a circle such that, if O is the centre of the circle, $\angle AOB = 110°$; *it is not a complete circle.*

In trying to discover what the locus is in any given problem it may be possible to visualise the path along which the object moves, as in the case of the seconds-hand considered above, but it is often better to start by marking in a figure a number of possible positions of the object and then try to guess what straight lines or curves form the locus. In either case it is necessary to prove that the guess is correct; but it is often harder to discover what the locus is than to prove the result when discovered. Oral practice in guessing loci should therefore be given before proceeding to theoretical proofs.

### Example for Oral Discussion

A, B are two given points in a given plane and P is a point in the plane such that $\angle APB = 60°$. Find experimentally the aggregate of all the possible positions of P.

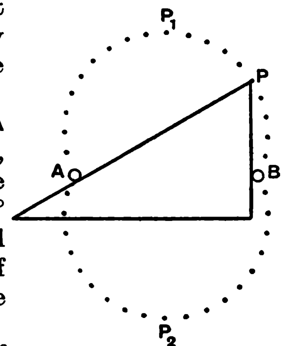


FIG. 417

Stick two pins into the paper at A and B, perpendicular to the paper, and slide your set-square between the pins so that the arms of the angle 60° of the set-square pass through A and B. Prick in the paper a number of possible positions of the vertex of the angle.

We see that the locus consists of two arcs of two distinct circles $AP_1B$ and $AP_2B$, not the whole circumference of one circle.

NOTE. The formal proof of this locus depends on a later theorem, Theorem 52, p. 332, for the present it must be regarded as an experimental result.

In this example, if the restriction that P lies in a given plane is removed, the aggregate of all its possible positions forms a *surface* which can be obtained by revolving the arc $AP_1B$ about the line AB as axis through 4 right angles.

**Definition.** The aggregate of all points whose positions are determined by a given law is called the **locus of points** subject to that law.

When there are an unlimited number of possible positions of a point P, all subject to some given law, we call any possible position of P a **variable point**, and it is often said that the variable point P moves in accordance with the given law and traces out the resulting locus. This form of words is not strictly accurate, because a point marks a position in a plane or in space and cannot move; its use is justified by regarding the phrase, "*variable point*," as meaning some small object, such as the tip of a pencil, which is free to move.

## EXERCISE 46 (Oral)

What are the loci described in Nos. 1–14?

1. A mark on the platform of a merry-go-round.

2. A small lump of lead dropped from your hand.

3. The tip of the pendulum of a clock.

4. The top of your head if you slide downstairs on a tea-tray.

[5] The tip of your nose in a swing.

[6] The centre of the wheel of an engine running along (i) a straight railway line, (ii) a circular railway line.

[7] A mark on a see-saw.

[8] The end of the chain by which a donkey is tethered to a post if the chain is kept fully stretched.

9. A mark on the top of a trap-door in the floor when the trap-door is opened to its full extent.

10. The tip of a man's nose on a moving staircase if he stands still from the time he steps on till the time he steps off.

11. The centre of the top of a box when the base of the box is made to slide about on the top of a table.

[12] The centre of a marble which rolls about inside a spherical bowl.

13. The highest point of the shade of an electric light hanging from the ceiling when the light swings about.

14. The right-angled corner of your set-square if you rotate it round the hypotenuse as axis.

15. A variable point P is due north of a fixed point A. Describe precisely the locus of P.

16. A variable point P is at a given distance from a given point A. What is the locus of P (i) if P lies in a plane through A, (ii) without this restriction?

17. A variable point P is at a given distance from a given line AB. What is the locus of P (i) if P lies in a plane through AB, (ii) without this restriction?

[18] A variable circle, centre P, of given radius, passes through a fixed point A. What is the locus of P (i) if P lies in a plane through A, (ii) without this restriction?

[19] A water-cart with pump attached can be wheeled to any part of a rectangular lawn surrounded by flower-beds. What is the boundary of the portion that can be watered?

[20] ABC is a piece of wire bent so that the straight portions AB and BC, whose lengths are given, are at right angles. The end A is fixed; what is the locus of C?

21. A penny is held flat on a table, and another penny, also flat on the table, is made to roll round it. What is the locus of the centre of the moving penny?

22. A circular disc is pivoted about a point on its rim. What is the locus of another marked point on the rim (i) if the disc can only turn in its own plane, (ii) without this restriction?

23. Four rods form a parallelogram ABCD; AB is fixed, but AD and BC can turn round A and B respectively. Find the locus of a marked point P on the rod CD. [Draw PQ parallel to DA to meet AB at Q.]

[24] A, B are fixed points; P is a variable point on a given circle, centre A; ABPQ is a parallelogram. Find the locus of Q. [Produce BA to C so that BA = AC; join CQ.]

25. ABC is a given triangle; BAPQ, CBQR are variable parallelograms. If P lies on a given circle, centre A, find the locus of R.

26. The base ABFE of a cubical box, see fig. 418, rests on the ground. The box is rolled over, without slipping, so that the face BFGC rests on the ground and then rolled over again so that the faces CGHD, DHEA in turn rest on the ground. Sketch the locus of A.

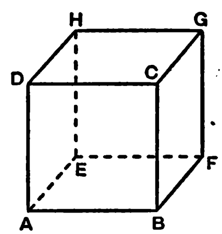


FIG. 418

[27] An equilateral triangle ABC rests in a vertical plane with AB in contact with a horizontal table. It is rotated in a vertical plane about B till BC is in contact with the table, and then rotated in a vertical plane about C till CA is in contact with the table. Sketch the locus of A.

[28] A car is moving in a straight line on level ground and comes to a raised obstacle with a level top which it climbs and passes over and returns to ground at the original level on the other side. Sketch the locus of the centre of one of the wheels.

29. A thin straight rod is 3 ft. long; P is a variable point such that its distance from the nearest point of the rod is always 2 ft. What is the complete locus of P (i) if P lies in a given plane through the rod, (ii) without this restriction?

*30. AB is a long string with a weight attached to B; the end A is free to slide on the rim of a fixed vertical circular ring and AB itself remains vertical. What is the locus of B?

*31. A and B are given points such that $AB = 5$ cm.; P is a variable point such that $PA < 4$ cm. and $PB < 3$ cm. What is the complete locus of P (i) if P lies in a given plane through AB, (ii) without this restriction?

*32. A long string is wound round a prism whose section is a regular hexagon. Describe the locus of the free end of the string when the string is unwound, being kept taut and in a plane perpendicular to the axis of the prism.

*33. A sphere rolls on the inside surface of a hollow circular cone. What is the locus of its centre?

*34. A variable point P lies on a variable line AQ which passes through a given point A and makes a given angle with a given plane. What is the locus of P?

*35. ABCD is a fixed rhombus; P is a variable point in the plane of ABCD such that $\angle APB = \angle APD$. Find out whether the diagonals and the diagonals produced form part of the locus of P. There is also a circular arc which belongs to the locus of P; sketch this arc in your figure by using experimental methods.

*36. ABCD is a square; P is a variable point inside the square such that the sum of its distances from AB and AD is equal to a side of the square. Use experimental methods to find the locus of P and then prove your result is correct. [Use squared paper.]

*37. A ladder 10 ft. long rests with one end against a vertical wall and the other end on a horizontal floor. If the ladder slips down, remaining in a plane perpendicular to the wall, find the locus of its mid-point.

*38. A circular cone rolls on a plane. What is the locus of the centre of the base of the cone?

*39. A solid, consisting of two unequal spheres glued together, rolls on a plane. What are the loci of the centres of the spheres?

In solving problems on loci, there are in general three stages:

(1) The locus must be discovered.

Sometimes this can be done experimentally. But when this is impracticable, it is necessary to reduce the problem to one of the standard locus theorems.

(2) It must be proved that the position of every point on the specified locus satisfies the given law.

(3) It must be proved that every point whose position satisfies the given law lies on the specified locus.

From theorems which will be discussed at a later stage dealing with areas, angle properties of a circle and similar figures, useful standard locus-theorems can be deduced.

At this stage, we consider two locus-thorems which have many important practical and theoretical applications.

**I. The locus of points which are equidistant from two given points is the perpendicular bisector of the straight line joining the given points.**

*Abbreviation for reference:* Perp. bisector locus.

**II. The locus of points which are equidistant from two given intersecting straight lines is the pair of lines which bisect the angles between the given lines.**

*Abbreviation for reference:* Angle bisector locus.

The enunciation of each theorem specifies the locus; the *complete* proof of each theorem involves the proving of two distinct statements, either of which is the converse of the other, corresponding to stages (2) and (3) enumerated above.

The first of these two locus theorems is established by the two theorems, 29 (1), 29 (2), pp. 200, 201, which deal with stage (2) and stage (3) respectively. The second of these locus theorems is established by the two theorems, 32 (1), 32 (2), pp. 208, 209.

### THEOREM 29 (1)

**Any point on the perpendicular bisector of the line joining two given points is equidistant from the given points.**

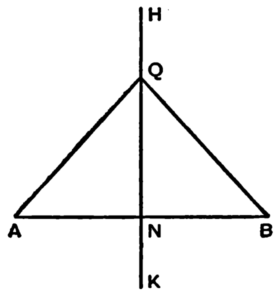



FIG. 419

**Given** two points A, B and any point Q on the perpendicular bisector HK of AB.

**To prove that** QA = QB.

**Construction.** Let HK cut AB at N.

**Proof.** In $\triangle$s ANQ, BNQ,

AN = BN *given,*

QN = QN,

$\angle$ANQ = $\angle$BNQ *rt. $\angle$s, given.*

$\therefore$ $\triangle$s $\genfrac{}{}{0pt}{}{\text{ANQ}}{\text{BNQ}}$ are congruent SAS.

$\therefore$ QA = QB.

## THEOREM 29 (2)

**A point which is equidistant from two given points lies on the perpendicular bisector of the straight line joining the given points.**

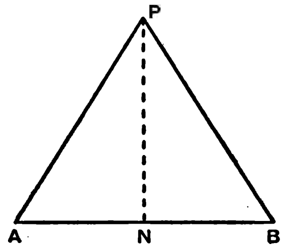



FIG. 420

**Given** two points A, B and a point P such that PA = PB.

**To prove that** P lies on the perpendicular bisector of AB.

**Construction.**

Let N be the mid-point of AB.
Join PN.

**Proof.** In △s ANP, BNP,

| | |
|---|---|
| AN = BN | *constr.*, |
| PA = PB | *given*, |
| PN = PN. | |

∴ △s $\begin{matrix}\text{ANP}\\\text{BNP}\end{matrix}$ are congruent SSS.

∴ ∠ANP = ∠BNP,

but these are adjacent ∠s on a straight line, therefore each is a right angle.

∴ PN is perpendicular to AB and bisects it,
∴ P lies on the perpendicular bisector of AB.

*Abbreviation for reference:* Perp. bisector locus.

### Examples for Oral Discussion

**1. The perpendicular bisectors of the sides AB, AC of a triangle ABC meet at O. Prove that**

**(1) OA = OB = OC;**

**(2) the perpendicular bisector of BC passes through O.**

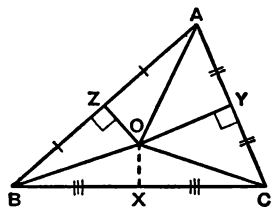



FIG. 421

(i) Explain why $\triangle$OZA $\equiv$ $\triangle$OZB; then complete the proof of (1).

(ii) If X is the mid-point of BC, explain why $\triangle$OXB $\equiv$ $\triangle$OXC; then complete the proof of (2).

This argument provides a proof of the theorem that the perpendicular bisectors of a triangle are concurrent. The proof may be abbreviated, see Theorem 30, by using the locus theorem just established.

Since OA = OB = OC, the circle, centre O, radius OA, passes through A, B, C. It is called the circumcircle of $\triangle$ABC, and O is called the **circumcentre**; the radius of this circle is called the **circumradius.**

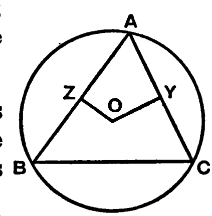



FIG. 422

**2. If A, B, C are any three given points which do not lie on a straight line, one and only one circle can be drawn to pass through them.**

Perform the necessary construction and prove that the statement is correct.

**3. If AD, BE, CF are the altitudes of any triangle ABC, i.e. the perpendiculars from the vertices to the opposite sides, prove that AD, BE, CF are concurrent.**

Through A, B, C draw lines parallel to BC, CA, AB respectively to form the triangle PQR.

(i) Explain why BC = AR and prove that AQ = AR.

(ii) Explain why AD is the perpendicular bisector of QR.

(iii) What can you say about BE and CF?

Complete the proof.

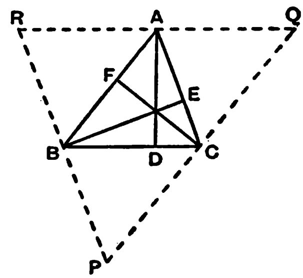



FIG. 423

The point of intersection of the altitudes is called the **orthocentre** of the triangle.

In fig. 423, the triangle DEF, whose vertices are the feet of the altitudes, is called the **pedal triangle** of the triangle ABC.

If you draw an *obtuse-angled* triangle, you will see that its orthocentre lies *outside* the triangle.

NOTE. The circum*centre* of △ ABC is the centre of the circle which passes through ABC, but the "ortho*centre*" is not the centre of any important circle associated with △ ABC. The word *centre* is often used to denote the common point of intersection of three or more concurrent straight lines. Thus the common point of intersection of the medians of a triangle (see p. 192) might be called the "median-centre" of the triangle, although actually it is called the centroid.

**Notation.** If corresponding points are taken on the sides of a triangle ABC, *e.g.* mid-points, feet of altitudes, etc., they should be denoted by consecutive letters of the alphabet, *e.g.* X, Y, Z or D, E, F, etc., see figs. 421, 423, with X on the side opposite A, Y opposite B, Z opposite C, and similarly for D, E, F. It is easier both to write out an argument and to follow it if this is done.

## THEOREM 30

**The perpendicular bisectors of the three sides of a triangle are concurrent.**

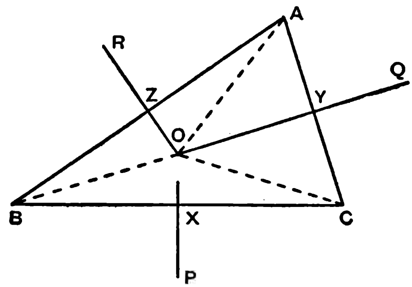



FIG. 424

**Given** a triangle ABC and the perpendicular bisectors PX, QY, RZ of BC, CA, AB.

**To prove that** PX, QY, RZ are concurrent.

**Construction.**

Let QY meet RZ at O.
Join OA, OB, OC.

**Proof.** Since O lies on the perpendicular bisector of AB,

$$OA = OB.$$

Since O lies on the perpendicular bisector of AC,

$$OA = OC,$$

$$\therefore\ OB = OC.$$

$\therefore$ O is equidistant from the points B, C;

$\therefore$ O lies on the perpendicular bisector PX of BC.

$\therefore$ PX, QY, RZ meet at O.

*Abbreviation for reference:* Circumcentre theorem.

## THEOREM 31

**The altitudes of a triangle are concurrent.**

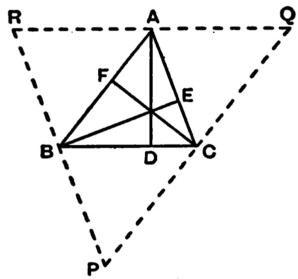



FIG. 425

**Given** a triangle ABC, and the perpendiculars AD, BE, CF from A, B, C to BC, CA, AB.

**To prove that** AD, BE, CF are concurrent.

**Construction.** Through A, B, C draw straight lines parallel to BC, CA, AB respectively, to form the triangle PQR.

**Proof.** BC is parallel to RA *constr.*,
CA is parallel to BR *constr.*,
∴ BCAR is a parallelogram,
∴ BC = RA *opp. sides* ||*gram.*

Similarly, since BCQA is a parallelogram,
BC = AQ.
∴ RA = AQ.

Since AD is perpendicular to BC *given*,
and since BC is parallel to RQ *constr.*,
AD is perpendicular to RQ.
∴ AD is the perpendicular bisector of RQ.

Similarly, BE, CF are perpendicular bisectors of RP, PQ.
But the perpendicular bisectors of the sides of △PQR are concurrent,
∴ AD, BE, CF are concurrent.

*Abbreviation for reference:* Orthocentre theorem.

## EXERCISE 47

1. Draw a triangle ABC in which BC = 7 cm., CA = 6 cm., AB = 5 cm. Construct the circle which passes through A, B, C and measure its radius.

[2] Given an arc of a circle, show how to construct the centre of the circle.

3. Draw a triangle ABC in which $\angle$B > $\angle$C; *construct in the simplest way* a point P on AC such that $\angle$PBC = $\angle$C.

[4] Draw a triangle ABC in which $\angle$A is obtuse. Construct a point P on BA produced such that PB − PC = AB.

5. Draw a quadrilateral ABCD in which AB is not parallel to DC. Construct a point P such that PA = PB and PC = PD.

[6] Draw a triangle XYZ and construct a point P such that PX = PZ and XP is perpendicular to YZ.

7. Draw a triangle ABC in which AB = 2 in., BC = 3 in., $\angle$B = 60°. Find a point P on CB produced such that PC − PA = $1\frac{1}{2}$ in. Measure PC. [Find two points from which P is equidistant.]

8. Given two points A, B and a line CD, construct a circle to pass through A and B and have its centre on CD. Is this always possible?

[9] Given a circle and two points A, B inside it. Construct a circle to pass through A, B and have its centre on the circumference of the given circle. Is there more than one solution?

10. In $\triangle$ABC, AB = AC. If the perpendicular bisector of AB cuts BC, or BC produced, at X, prove that $\angle$AXB = $\angle$BAC.

[11] Two circles, centres A, B, cut at H and K. Prove that AB bisects HK at right angles.

12. ABC is a triangle right-angled at C; the perpendicular bisector of AC cuts AB at K. Prove that KA = KB = KC.

Where is the circumcentre of a right-angled triangle?

[13] A, B are given points; APBQ is a variable rhombus. Find the locus of P.

14. A, B are given points; AQ is a variable line; P is the image of B in AQ, *i.e.* AQ is the perpendicular bisector of BP. Find the locus of P.

15. A, B, C, D are four points on the circumference of a circle. Prove that the perpendicular bisectors of AB, AC, AD, BC, BD, CD are concurrent.

[16] AB is the longer of the two parallel sides AB, DC of the trapezium ABCD. If P is a point on AB such that DP = DA and CP = CB, prove that AB = 2CD.

17. In fig. 426, RP, RQ are the bisectors of the equal angles APB, AQB. If RP = RQ, prove that A, R, B lie on a straight line. [Join PQ.]

[18] The angle BAC of $\triangle$ ABC is obtuse, and the perpendicular bisectors of AB, AC cut BC at H, K. Prove that

$$\angle HAK = 2\angle BAC - 180°.$$

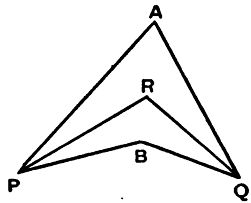


FIG. 426

*19. The diagonals of the quadrilateral ABCD cut at K. Circles are drawn through A, K, B; B, K, C; C, K, D; D, K, A. Prove that their centres are the vertices of a parallelogram.

*20. In fig. 427, AB and AC are given straight lines; APQR is a variable parallelogram of given perimeter. Prove that the locus of Q is part of a straight line. What is the complete locus if P, R can also lie on BA produced, CA produced? [Take K on AB so that PK = PQ and prove that KQ is in a fixed direction.]

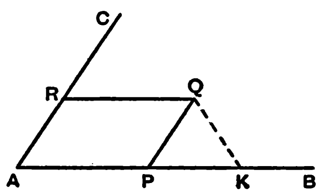


FIG. 427

[*The following examples depend on the orthocentre property.*]

[21] Where is the orthocentre of a right-angled triangle?

22. If D is the orthocentre of $\triangle$ ABC, prove that A is the orthocentre of $\triangle$ DBC.

[23] X, Y, Z are the mid-points of the sides of $\triangle$ ABC; prove that the orthocentre of $\triangle$ XYZ is the circumcentre of $\triangle$ ABC.

24. If H is the orthocentre of $\triangle$ ABC, prove that the angles BHC, BAC are equal or supplementary. [$\triangle$ ABC may be acute-angled or obtuse-angled.]

25. In $\triangle$ ABC, $\angle$ A = 45°; H is the orthocentre of $\triangle$ ABC and CH cuts AB at F; prove that BF = FH.

*26. Q is a point inside the parallelogram ABCD such that $\angle$QBC and $\angle$QDC are right angles; prove that AQ is perpendicular to BD.

## THEOREM 32 (1)

**A point which lies on the bisector of a given angle is equidistant from the arms of that angle.**

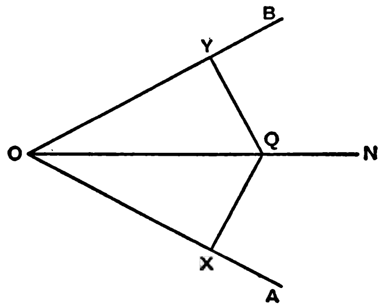



FIG. 428

**Given** an angle AOB,
a point Q on the bisector ON of $\angle$AOB,
and the perpendiculars QX, QY from Q to OA, OB.

**To prove that** QX = QY.

**Proof.** In $\triangle$s QXO, QYO,

$\angle$QOX = $\angle$QOY *given,*

$\angle$QXO = $\angle$QYO *rt.* $\angle$*s, given,*

QO = QO.

$\therefore$ $\triangle$s $\begin{matrix}QXO\\QYO\end{matrix}$ are congruent AAS.

$\therefore$ QX = QY.

## THEOREM 32 (2)

**A point which is equidistant from two intersecting straight lines lies on one of the lines which bisects the angles between the given lines.**

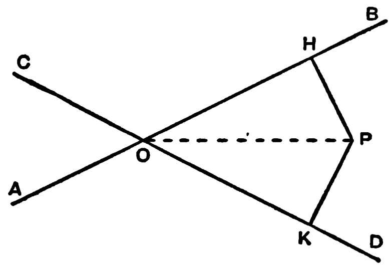



FIG. 429

**Given** two straight lines AOB, COD and a point P such that the perpendiculars PH, PK from P to AB, CD are equal.

**To prove that** P lies on the bisector of one of the angles between AOB, COD.

**Construction.** Join OP.

**Proof.** Suppose that P lies within the angle BOD.

In $\triangle$s PHO, PKO,

$\angle$PHO = $\angle$PKO *rt. $\angle$s, given,*

PH = PK *given,*

PO = PO.

$\therefore$ $\triangle$s $\begin{matrix}\text{PHO}\\\text{PKO}\end{matrix}$ are congruent RHS.

$\therefore$ $\angle$POH = $\angle$POK.

$\therefore$ P lies on the bisector of $\angle$BOD.

In the same way, it may be proved that, if P lies within any one of the angles BOC, COA, AOD, it lies on the bisector of that angle.

*Abbreviation for reference:* $\angle$ bisector locus.

## Examples for Oral Discussion

**1. The bisectors of $\angle$B, $\angle$C of $\triangle$ABC meet at I; IP, IQ, IR are the perpendiculars from I to BC, CA, AB. Prove that**

(i) IP = IQ = IR. (ii) IA bisects $\angle$BAC.

(i) Explain why $\triangle$IPB $\equiv$ $\triangle$IRB; then complete the proof of (i).

(ii) Join IA and explain why $\triangle$IRA $\equiv$ $\triangle$IQA; then complete the proof of (ii).

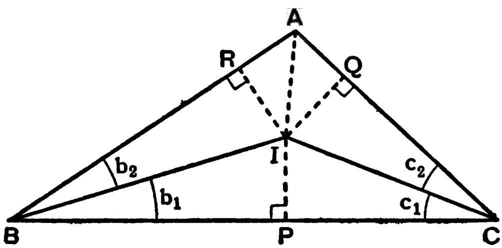


FIG. 430

This argument provides a proof of the theorem that **the bisectors of the three angles of any triangle are concurrent.** The proof may be abbreviated, see Theorem 33, by using the locus theorem just established.

Since IP = IQ = IR, the circle, centre I, radius IP, passes through P, Q, R.

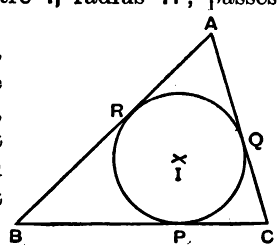


FIG. 431

Also since the least distance of a point from a straight line is the perpendicular distance, the circle, centre I, radius IP, does not meet BC at any point except P, and similarly does not meet CA, AB at any points except Q, R.

This circle is said to **touch** BC, CA, AB and is called the **inscribed circle** of the triangle ABC; its centre I is called the **in-centre,** and the radius of the circle is called the **in-radius.**

**External Bisector of an Angle.** If one arm AB of an angle ABC is produced to D, the bisector of the angle DBC is called the *external bisector* of the angle ABC.

**2. The external bisectors of $\angle$B, $\angle$C of $\triangle$ABC meet at $I_1$. If $I_1P_1$, $I_1Q_1$, $I_1R_1$ are the perpendiculars from $I_1$ to BC, AC produced, AB produced, prove that**

**(i) $I_1P_1 = I_1Q_1 = I_1R_1$;**

**(ii) $I_1A$ bisects $\angle$BAC;**

**(iii) the circle, centre $I_1$, radius $I_1P_1$, touches BC, AB produced, and AC produced.**

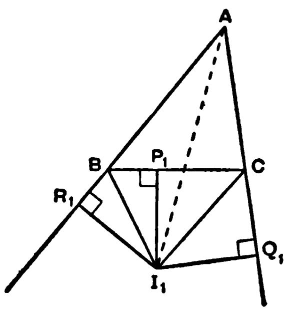



FIG. 432

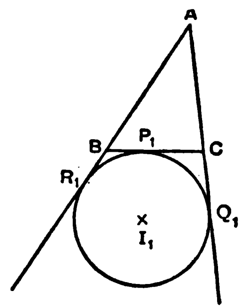



FIG. 433

(i) Explain why $\triangle I_1P_1B \equiv \triangle I_1R_1B$; then complete the proof of (i).

(ii) Join $I_1A$ and explain why $\triangle I_1R_1A \equiv \triangle I_1Q_1A$; complete the proof of (ii).

(iii) Explain why the circle, centre $I_1$, radius $I_1P_1$, meets BC at no point except $P_1$, and complete the proof.

The circle which touches BC, AB produced, AC produced, see fig. 433, is called an escribed circle of $\triangle$ABC, and its centre $I_1$ is called an **ex-centre**.

There are *three* escribed circles of any triangle; the circle in fig. 433 is said to be escribed to BC. Similarly, two other circles can be constructed, escribed to AB and escribed to AC respectively.

NOTE. The proof of example 2 shows that the *external* bisectors of any two angles of a triangle and the *internal* bisector of the third angle are concurrent.

## Intersection of Loci

If the position of a point is given by two distinct conditions, it may be possible to construct the two corresponding loci and so fix the position, or the several possible positions, of the point by taking the intersection of these lines or curves.

For example, we have just constructed the possible positions of points which are equidistant from the three sides of the triangle ABC:

(i) the locus of points equidistant from BA, BC consists of the internal and external bisector of $\angle$ABC;

(ii) the locus of points equidistant from CA, CB consists of the internal and external bisector of $\angle$ACB.

These two loci, two pairs of straight lines, intersect at four points, the in-centre and the three ex-centres of $\triangle$ABC. Each of these points is equidistant from the lines which form the sides of $\triangle$ABC.

The use of the intersection of loci was illustrated by the proof of Theorem 30 and is again illustrated by the proof of Theorem 33. The reader should employ the same method to prove that the *external* bisectors of two angles of a triangle and the *internal* bisector of the third angle are concurrent.

## THEOREM 33

**The internal bisectors of the three angles of a triangle are concurrent.**

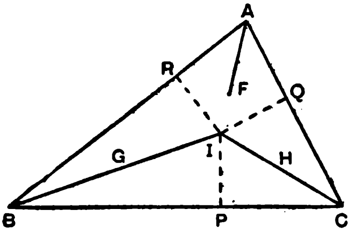



FIG. 434

**Given** a triangle ABC and the internal bisectors AF, BG, CH of ∠A, ∠B, ∠C.

**To prove that** AF, BG, CH are concurrent.

**Construction.** Let BG, CH meet at I.
From I draw the perpendiculars IP, IQ, IR to BC, CA, AB.

**Proof.** Since I lies on the bisector of ∠ABC,
I is equidistant from the straight lines BC, BA,

$$\therefore \; IP = IR.$$

Since I lies on the bisector of ∠ACB,
I is equidistant from the straight lines CB, CA,

$$\therefore \; IP = IQ.$$
$$\therefore \; IQ = IR.$$

∴ I is equidistant from the straight lines AC, AB and lies within the angle BAC,
∴ I lies on the internal bisector AF of ∠BAC;

$$\therefore \; AF, \; BG, \; CH \text{ meet at } I.$$

*Abbreviation for reference:* In-centre theorem.

## EXERCISE 48

[*Give all possible answers of the required constructions.*]

**1.** ABC is an equilateral triangle of side 3 in. What is the locus of (i) points distant 2 in. from A, (ii) points distant 1 in. from BC? Construct a point P 2 in. from A and 1 in. from BC.

[2] ABC is an equilateral triangle of side 4 cm. Find points on AC and AC produced which are 2 cm. from BC.

3. ABC is an equilateral triangle of side 4 cm. Construct a point P, 2 cm. from AB and 3 cm. from AC.

[4] ABC is an equilateral triangle of side 4 cm. Construct a point P, 2 cm. from AC and equidistant from BA and BC.

5. Draw $\triangle$ABC such that AB = 4 cm., BC = 5 cm., CA = 4·5 cm. Construct a point P which is equidistant from CA, CB and is 3 cm. from B.

6. Draw two straight lines AOB, COD cutting at an angle of 60°. Sketch the locus of a variable point P whose distance is always 2 cm. from one of the given lines.

What part of this locus must be omitted if P is never less than 2 cm. from either of the given lines? Rub out this part.

[7] ABCD is a rectangle such that AB = 5 cm., BC = 4 cm. Sketch the locus of points whose distances are always 1 cm. from some side of the rectangle, either inside or outside ABCD. What part of this locus must be omitted if they are never less than 1 cm. from any side of ABCD? Rub it out.

8. ABC is a triangle such that AB = 4 cm., BC = 5 cm., CA = 6 cm. Find the locus of points whose distances are always 1 cm. from some side of the triangle and never less than 1 cm. from any side, either inside or outside ABC. [Use the "*draw and rub out*" method indicated in Nos. 6, 7.]

[9] A, B are two given points such that AB = 5 cm. Find the locus of points which are always 4 cm. from one of the points A, B and never less than 4 cm. from the other.

10. Draw a circle, radius 4 cm., and draw a straight line ABCD cutting the circle at B, C such that BC = 4 cm. Find the locus of points which lie on the line AD and are always more than 1 cm. from the nearest point of the circumference of the given circle.

11. ABC is an equilateral triangle of side 1 in. Sketch the locus of points in the plane of ABC whose distances are always 1 in. from some one of the three points A, B, C, and never less than 1 in. from the other two.

[12] ABCD is a square of side 6 cm. Sketch the locus of points which are equidistant from the lines AB, AD, and distant not less than 2 cm. from any side of the square.

13. Draw any triangle ABC. Construct points on BC and BC produced which are equidistant from AB and AC.

[14] Draw any triangle ABC. Construct a point P which is equidistant from AB and AC, and also equidistant from B and C. Construct also the circumcircle of $\triangle$ABC. Does P lie on its circumference?

15. Draw a triangle ABC such that BC = 5 cm., CA = 4 cm., AB = 6 cm. Construct the four points which are equidistant from the three sides of the triangle. Construct also the in-circle and as much of the three escribed circles as there is room for on your paper.

[16] Draw a parallelogram ABCD. Construct the two points which are equidistant from the three sides AB, BC, CD, and construct the two circles which touch these three lines.

17. The diagonals of the parallelogram ABCD cut at K. If K is equidistant from AB and AD, prove that ABCD is a rhombus.

18. Construct a triangle ABC such that BC = 5 cm., the median AA′ = 4 cm., and the perpendicular AD from A to BC is 3 cm. State shortly your method.

19. Draw on squared paper, with 1 inch as unit on each axis, the locus of points whose co-ordinates $(x, y)$ are subject to the following laws:—

(i) $y = 2x$; (ii) $y = \frac{1}{2}x + 1$;
(iii) $y = 3 - x$; (iv) $4y + 5x = 16$;
(v) $y = \frac{1}{5}x^2$; (vi) $5y = 2x^2 - 7x + 7$.

*Nos.* **20–23** *should be worked on squared paper.*

**20.** Take a point S 2 in. from the lowest main line XK on the squared paper and on a main *central* line XS up the paper. A variable point P is such that its distance from S is equal to its distance from the line XK. Construct a number of possible positions of P (on both sides of SX) and then draw a free-hand curve to represent the locus of P. This curve is called a **parabola.**

Taking X as origin and XK as $x$-axis and the unit on each axis as 1 in., draw on the same figure the graph of $\frac{1}{8}x^2+1$.

**21.** Take a point S 2 in. from the lowest main line XK on the squared paper and on a main *central* line XS up the paper. A variable point P is such that its distance from S is three-fifths of its distance from XK. Construct a number of possible positions of P (on both sides of SX) and then draw a free-hand curve to represent the locus of P. This curve is called an **ellipse.** How high up, how low down can the curve go?

[22] In the figure for No. 21, mark a point S′ on XS produced such that $S'X = 4{\cdot}25$ in. A variable point P is such that $PS + PS' = 3{\cdot}75$ in. Draw the curve which represents the locus of P and compare the result with the locus in No. 21. [If you have a loop of thread 6 in. long and put pins in the paper at S and S′ you can trace this locus very quickly.]

**23.** Take a point S 1 in. below the *highest* main line on the squared paper and on a main central line SX. Call the line, which is $2\frac{1}{2}$ in. below the highest main line, XK. A variable point P is such that its distance from S is twice its distance from XK. Construct a number of possible positions of P (on both sides of SX) and then draw free-hand the locus of P. There are two distinct branches of the curve, one on each side of XK. The curve is called a **hyperbola.**

[24] Take two points A, B, 6 cm. apart. A variable point P is such that $PA = 2PB$. Draw the curve which represents the locus of P. On AB produced take a point C such that $BC = 2$ cm., and add to your figure the circle, centre C, radius 4 cm.

*For Nos.* 25–27, *tracing paper should be used.*

**25.** Draw two lines OA, OB cutting at right angles. P, Q are variable points on OA, OB such that $PQ = 8$ cm.; R, S are points on PQ such that R is the mid-point of PQ, and $PS = 3$ cm. Draw a line PSRQ on the tracing paper so that $PS = 3$ cm., $PR = 4$ cm., $PQ = 8$ cm. and, by pricking through, obtain a number of possible positions of R, S. Then sketch the locus (i) of R, (ii) of S.

[26] Mark on your paper two points A, B, 5 cm. apart. Draw on tracing paper two lines cutting at an angle of 50°. By pricking through, find the locus of a point P such that $\angle APB = 50°$.

*27. Draw on your paper a circle, diameter AB, such that AB = 4 cm. A variable line through A cuts the circle again at P and is produced to Q so that PQ = 4 cm.; also PA is produced to Q′ so that PQ′ = 4 cm. Draw on tracing paper a straight line LQ′PQN such that Q′P = PQ = 4 cm. and, by pricking through, obtain a number of possible positions of Q, Q′, and sketch the locus. The curve is called a **cardioid.**

*28. Draw two perpendicular lines AB, AC. P is a variable point which is 2 cm. nearer to AB than to AC. Find the locus of P. [Draw a line DE parallel to AB such that P is equidistant from AC and DE.]

*29. ABC is an equilateral triangle of side 6 cm. Construct a point P on the line AB which is 2 cm. nearer to BC than to AC. Is there more than one possible position?

*30. ABC is a triangle in which AB > AC. The perpendicular bisector of BC meets the bisector of $\angle BAC$ at P; PX, PY are the perpendiculars from P to AB, AC produced. Prove that BX = CY. [Join PB, PC.]

*31. ABCD is a given quadrilateral. Construct a point P which is equidistant from the lines AD, BC, and also equidistant from the lines AB, CD. What is the greatest number of possible positions of P? How many positions of P are there if ABCD is a parallelogram? Prove that there are not more than two possible positions of P if $\angle A + \angle C$ is equal to two right angles.

*32. A variable line cuts two given lines AB, CD at P, Q. If the bisectors of $\angle BPQ$, $\angle DQP$ meet at R, find the locus of R.

*33. AB and CD are two given lines which when produced meet off the paper. Construct that portion of the line bisecting the angle between AB and CD which lies on the paper. [Compare No. 32.]

*34. In fig. 433, p. 211, if CB is produced to H and if BC is produced to K so that HB = BA and KC = CA, prove that $I_1A = I_1H = I_1K$.

*35. If I is the in-centre and if $I_1$, $I_2$, $I_3$ are the ex-centres of a triangle ABC, prove that the line joining any two of the points I, $I_1$, $I_2$, $I_3$ is perpendicular to the line joining the other two.

# REVISION PAPERS 1–34

## REVISION PAPERS 1–8 (Theorems 1–9)

[*Angle properties of parallels, triangles, polygons.*]

### 1

1. What angle is equal to (i) $\frac{1}{5}$ of its supplement?
   (ii) $\frac{1}{5}$ of its complement?

2. ABCD is a straight line such that $AB = \frac{1}{3}AC = \frac{1}{5}AD$. If $CD = 2$ in., calculate the distance between the mid-points of AB and AD.

3. In fig. 435, arrows indicate lines are given parallel. Calculate the angles $a$ and $b$. Give reasons.

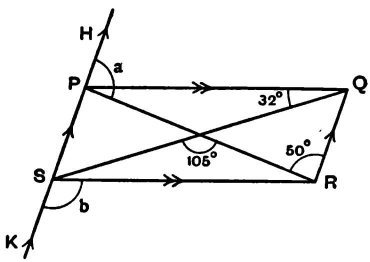



FIG. 435

4. In $\triangle ABC$, $\angle A = 74°$, $\angle B = 52°$; AB is produced to D, BC is produced to E. Calculate the acute angle between the bisectors of $\angle DBC$, $\angle ECA$.

### 2

1. (i) Find the obtuse angle between the directions N. 37° E., S. 54° W.
   (ii) Find the reflex angle between the directions N.W., W.

2. A line AB, 3 in. long, is produced to points P, Q such that $AP = 4BP$ and $AQ = 3BQ$. Find the length of PQ.

3. Calculate the angle $c$ in fig. 436.

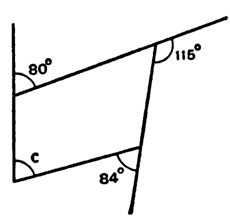



FIG. 436

4. D is a point on the side BC of $\triangle ABC$ such that $\angle CAD = \angle ABC$. Prove that $\angle ADC = \angle BAC$.

## 3

1. ACB and DCE are two straight lines.

   (i) If $\angle$ACE $=4\angle$BCE, calculate $\angle$BCE.
   
   (ii) If $\angle$ACE exceeds $\angle$BCE by $90°$, calculate $\angle$BCD.

2. ABC is a straight line; D is the mid-point of BC. Prove that AB + AC $=2$AD.

3. AD is the perpendicular from A to the side BC of $\triangle$ABC. Given that AD $=4$ cm., $\angle$B $=55°$, $\angle$C $=65°$, draw $\triangle$ABC and measure BC.

4. X is a point inside $\triangle$ABC such that $\angle$XAB $=\angle$XCA. Prove that $\angle$AXC $+\angle$BAC equals 2 right angles.

## 4

1. A is due east of B; P is N. 17° W. of A and N. 29° E. of B. Calculate $\angle$APB.

2. The bisector of $\angle$A in $\triangle$ABC meets BC at P. Given that AP $=5$ cm., $\angle$B $=28°$, $\angle$C $=68°$, draw $\triangle$ABC and measure BC.

3. Fig. 437 represents a "Pentagram," that is, the inner figure is a regular pentagon. Calculate the angle $a$.

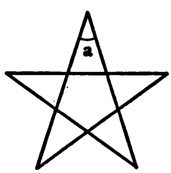



FIG. 437

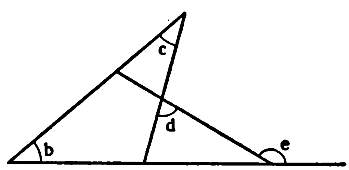



FIG. 438

4. In fig. 438, find $e$ in terms of $b$, $c$, $d$.

## 5

1. In fig. 439, not drawn accurately,

   (i) find three points which are collinear;
   
   (ii) find the angle $b$ if R, A, N are collinear;
   
   (iii) find what points are collinear if $b=2x°+5°$.

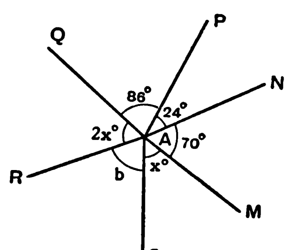



FIG. 439

2. ABCD is a quadrilateral with its opposite sides parallel, and AC bisects $\angle$BAD. If $\angle$BAD $=2y°$ and $\angle$ABC $=3y°$, calculate $\angle$ACB.

3. The sum of the angles of a polygon is 12 right angles. Find the number of sides of the polygon.

4. In fig. 440, $\angle$ACD $=\angle$ABC, and CP bisects $\angle$BCD. Prove that $\angle$APC $=\angle$ACP.

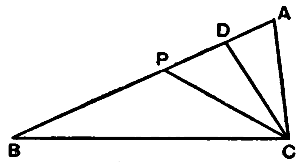


FIG. 440

## 6

1. It requires 4 complete turns of the handle to wind up a bucket from the bottom of a well 24 ft. deep. Through what angle must the handle be turned to raise the bucket 10 ft.?

2. In fig. 441, AB is parallel to EF. Calculate the value of $x$. Give reasons.

3. In $\triangle$ABC, AD is the perpendicular from A to BC, and the bisector of $\angle$BAC cuts BC at P. If $\angle$B $=2x°$ and $\angle$C $=2y°$, find in terms of $x$ and $y$ (i) $\angle$BAP, (ii) $\angle$PAD.

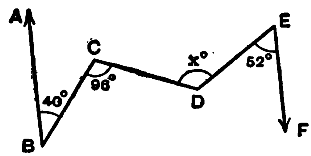


FIG. 441

4. The sides AD, BC of the quadrilateral ABCD are produced to P, Q respectively. If $\angle$DCQ $=\angle$A, prove that $\angle$PDC $=\angle$B.

## 7*

1. In fig. 442, ACB is a straight line. If $\angle$ACQ $=2\angle$QCT and if $\angle$BCR $=2\angle$RCT, find the size of $\angle$QCR.

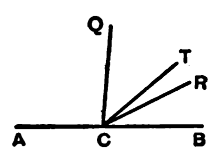


FIG. 442

2. In $\triangle$ABC, $\angle$A is greater than $\angle$B. If AN is the perpendicular from A to the line bisecting $\angle$ACB, prove that $\angle$BAN $=\frac{1}{2}(\angle$BAC $-\angle$ABC$)$.

3. The angles of a pentagon taken in order are $x+10$, $2x-10$, $2x$, $2x+10$, $x+50$, degrees. Find the value of $x$ and prove that two pairs of sides of the pentagon are parallel.

4. In fig. 443, PA, PB, RC, RD are the bisectors of the angles of the quadrilateral ABCD. Prove that $\angle$QPS and $\angle$QRS are supplementary.

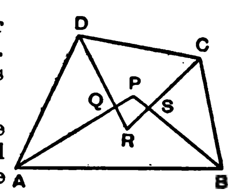


FIG. 443

## 8*

1. If the direction N. $2x°$ W. is the same as the direction S. $3x°$ W., find the value of $x$ and the true bearing of the direction (*i.e.* the angle which the direction makes with due north, measured clockwise).

2. In $\triangle$ ABC, the bisector of $\angle$ BAC meets BC at D and the bisector of $\angle$ ABC meets AC at E. If $\angle$ ADC $=79°$, $\angle$ BEC $=83°$, find $\angle$ ABC and $\angle$ ACB.

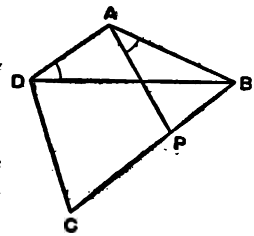


FIG. 444

3. Each angle of a regular polygon of $x$ sides is $\frac{4}{5}$ of each angle of a regular polygon of $y$ sides. Express $y$ in terms of $x$, and find any values of $x$ and $y$ which will fit.

4. In fig. 444, the angles DAB, DCB are supplementary and $\angle$ PAB $=\angle$ ADB. Prove that

$$\angle APB = \angle BDC.$$

## REVISION PAPERS 9–16 (Theorems 1–14)

[*Angle properties of triangles and polygons, congruence, isosceles triangles.*]

## 9

1. In $\triangle$ ABC, $\angle$ A $=44°$, $\angle$ B $=112°$. Calculate the acute angle between the bisectors of $\angle$ B and $\angle$ C.

2. In $\triangle$ ABC, AB $=$ AC and $\angle$ A $=20°$. D is a point on AC such that $\angle$ DBC $=60°$; prove that AD $=$ DB.

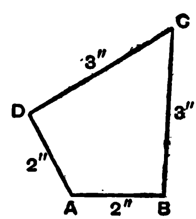


FIG. 445

3. In fig. 445, prove that $\angle$ ABC $=\angle$ ADC.

4. AB, DC are the parallel sides of the trapezium ABCD. If AD $=$ DC, prove that AC bisects $\angle$ BAD.

## 10

1. If the reflex angle AOB is four times the acute angle AOB, find the acute angle AOB.

2. In fig. 446, find $d$ in terms of $a$, $b$, $c$.

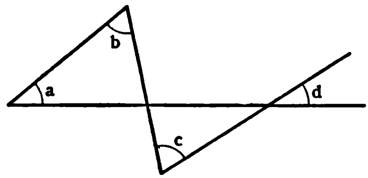



Fig. 446

3. In $\triangle$ABC, AB = AC; D is a point on AC such that AD = BD = BC. Calculate $\angle$BAC.

4. ABCD is a straight line such that AB = CD; K is a point outside the line such that KB = KC. Prove that KA = KD.

## 11

1. In $\triangle$ABC, $\angle$B = 35°, $\angle$C = 75°; the perpendiculars from B, C to AC, AB respectively cut at Q. Find $\angle$BQC.

2. ABC is an equilateral triangle; BC is produced to D so that BC = CD. Prove that $\angle$BAD is a right angle.

3. The bisector of $\angle$A of $\triangle$ABC cuts BC at D. Through C a line is drawn parallel to DA to meet BA produced at P. Prove that AP = AC.

4. If two circles have the same centre O, and if a straight line XABY is drawn cutting the inner circle at A, B and the outer circle at X, Y, prove that BX = AY.

## 12

1. The sum of one pair of angles of a triangle is 100°, and the difference of another pair is 60°. Prove that the triangle is isosceles.

2. If all the marked angles in fig. 447 are equal, find the size of each. Does the figure contain any pairs of parallel lines? Give reasons.

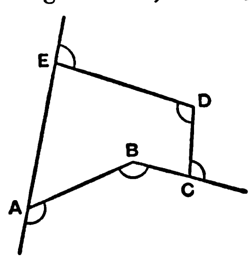



Fig. 447

3. In $\triangle$ABC, AB = AC; BC is produced to D so that CD = AB. Prove that $\angle$ABD = 2$\angle$ADB.

4. The diagonals of the quadrilateral ABCD cut at K. If AB = BC = CD and if $\angle$ABC = $\angle$BCD, prove that (i) KB = KC, (ii) $\angle$AKD = $\angle$ABC.

## 13

1. O is a point outside a straight line ABCD such that $OA = AB$, $OB = BC$, $OC = CD$; $\angle BOC = x°$; calculate $\angle OAD$ and $\angle ODA$ in terms of $x$.

2. In fig. 448, find $x$ in terms of $a$, $b$, $c$.

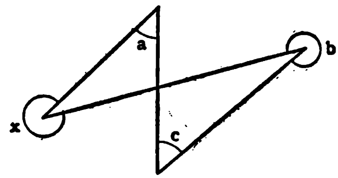


FIG. 448

3. In $\triangle ABC$, $AB = AC$; AB is produced to D so that $BD = BC$. Prove that $\angle ACD = 3\angle ADC$.

4. ACB is a straight line; ABX, ACY are equilateral triangles on opposite sides of AB. Prove that $CX = BY$.

## 14

1. In $\triangle ABC$, $AB = AC$; BC is produced to D so that $BD = BA$. If $\angle BAC = 2\angle CAD$, find $\angle ABC$.

2. Find the sum of the interior angles of a 15-sided polygon.

3. In fig. 449, $AB = AC$ and KAD is a straight line. Prove that $r - s = 2p$.

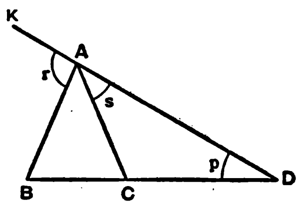


FIG. 449

4. In $\triangle ABC$, $AB = AC$ and $\angle A$ is a right angle. D is any point on CB; BH, CK are the perpendiculars from B, C to AD, produced if necessary. Prove that (i) $BH = AK$; (ii) the difference between BH and CK is equal to HK.

## 15*

1. In $\triangle ABC$, $\angle A = x°$, $\angle B = 3x°$, $\angle C = 5x°$. Find the value of $x$, and prove that a line BP can be drawn cutting AC at P such that $\triangle APB$ and $\triangle PCB$ are both isosceles.

2. The bisectors of $\angle B$, $\angle C$ of $\triangle ABC$ meet at I. If $\angle BIC = 135°$, prove that $\angle A$ is a right angle.

3. In fig. 450, arrows indicate that lines are given parallel. Prove that $a - b = c$.

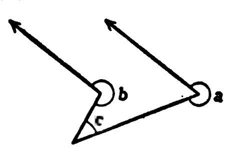


FIG. 450

4. O is a point inside an equilateral triangle ABC; OAP is an equilateral triangle such that O and P are on opposite sides of AB. Prove that $BP = OC$.

## 16*

1. In $\triangle ABC$, $\angle A = 115°$, $\angle C = 20°$; AD is the perpendicular from A to BC. Prove that $AD = DB$.

2. ABCDE is a regular pentagon; AC cuts BE at K. Find $\angle EKC$.

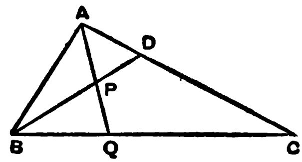



FIG. 451

3. In fig. 451, $\angle ABD = \angle C$, and APQ is the bisector of $\angle A$. Prove that $BP = BQ$.

4. In $\triangle ABC$, $\angle B = 90°$, $\angle C = 30°$ Prove that $AC = 2AB$.

## REVISION PAPERS 17–24 (Theorems 1–20)

[*Angles of polygon, congruence, isosceles triangles, parallelograms.*]

## 17

1. In $\triangle ABC$, $\angle A = 2\angle B$ and $\angle C - \angle B = 36°$. Prove that $\triangle ABC$ is isosceles.

2. ABCDE is a pentagon such that AB and DC when produced cut at right angles. If $\angle A = \angle D = 2\angle E$, find $\angle E$.

3. In $\triangle ABC$, $AB = AC$, and BC is less than AB. D is a point on BC produced such that $BD = BA$. Prove that $\angle ACD = 2\angle ADB$.

4. The diagonals of the parallelogram ABCD cut at K. Any line through K cuts AD, BC at P, Q. Prove that DPBQ is a parallelogram.

## 18

1. The angles of a quadrilateral, taken in order, are $x$, $x+20$, $x+30$, $x+50$, degrees. Find the value of $x$ and prove that the quadrilateral is a trapezium.

2. Draw the parallelogram ABCD in which AC $=3{\cdot}4$ in., BD $=4{\cdot}2$ in., BC $=2$ in. Measure AB.

3. In $\triangle$ABC, AB = AC. D is a point either on AC or on CA produced such that DB = BC. Prove that $\angle$DBC = $\angle$BAC. [Draw two figures, one in which $\angle$A is less than 60°, the other in which $\angle$A is greater than 60°.]

4. In fig. 452, BAC and PDQ are parallel lines. If AP, AQ are the bisectors of $\angle$BAD, $\angle$CAD, prove that PD = DQ.

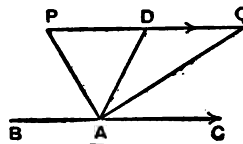


FIG. 452

## 19

1. Two equilateral triangles ABC, AYZ lie outside each other. If $\angle$CAY = 35°, find the acute angle at which BC and ZY intersect when produced.

2. In $\triangle$ABC, $\angle$C = 3$\angle$B. From AB a part AD is cut off equal to AC. Prove that CD = DB.

3. ABCD is a square; AXB is an equilateral triangle outside the square. Prove that $\angle$ACX = $\frac{1}{2}\angle$ABX.

4. ABCD is a parallelogram; BP, DQ are two parallel lines cutting AC at P, Q. Prove that BQ is parallel to PD.

## 20

1. State (without proof) the name of a quadrilateral in which
   (i) the diagonals bisect each other;
   (ii) the diagonals bisect each other at right angles;
   (iii) the diagonals are equal and bisect each other;
   (iv) all four sides are equal;
   (v) all four angles are equal.

2. AB, BC, CD, DE are four consecutive sides of a regular 20-sided polygon. Find the acute angle at which AB and ED intersect when produced.

3. ABCD is a parallelogram. If the bisector of $\angle$BCD cuts AD at its mid-point K, prove that KB bisects $\angle$ABC.

4. P is any point on the side AB of the square ABCD. The line from A perpendicular to DP cuts BC at Q. Prove that DP = AQ.

## 21

1. ABCDEFGH is a regular octagon. Find the acute angle at which AD cuts BF.

2. Draw a trapezium ABCD in which AB and DC are the parallel sides, given that AB = 7·5 cm., BC = 3 cm., CD = 3·5 cm., DA = 3·8 cm. Measure $\angle$BAD.

3. In $\triangle$ABC, AB = AC. P is any point on BC, and H, K are points on AB, AC such that AHPK is a parallelogram. Prove that PH + PK = AB.

4. ABCD is a square. The bisector of $\angle$BCA cuts AB at P; PQ is the perpendicular from P to AC. Prove that PB, PQ, AQ are all equal.

## 22

1. In $\triangle$ABC, $\angle$A = 55°, $\angle$C = 35°; K is a point on AC such that KB = KA. Prove that KB = $\frac{1}{2}$AC.

2. The diagonals of the rectangle ABCD cut at X. If AC = 5 cm. and $\angle$AXB = 110°, draw the rectangle and measure AB.

3. ABCD is a quadrilateral. If the bisectors of $\angle$ABC, $\angle$ADC are parallel, prove that $\angle$A = $\angle$C.

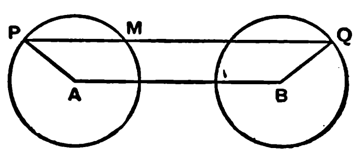



FIG. 453

4. In fig. 453, A, B are the centres of two equal circles. If $\angle$PAB = $\angle$QBA, prove that (i) $\angle$APQ = $\angle$BQP, (ii) QM = AB.

## 23*

1. In $\triangle ABC$, $\angle A = 2\angle B$; P is a point on BC, and Q is a point on AC, such that $AB = AP = PQ = QC$. Find $\angle C$.

2. In fig. 454, $AB = AC$, $BP = BQ$ and $PA = PR$. Prove that $\angle B = 3\angle A$.

3. The bisectors of $\angle B$, $\angle D$ of the quadrilateral ABCD meet at a point K inside ABCD; BK is produced to N. Prove that the difference of $\angle A$, $\angle C$ is equal to $2\angle NKD$.

Can you prove a corresponding result in the case when K lies outside ABCD?

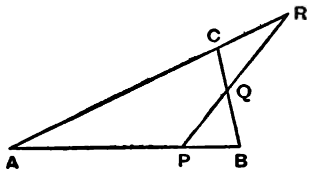


FIG. 454

4. ABCD is a square whose diagonals cut at K; P is a point on DK between D and K. The perpendicular from B to AP cuts AK at Q. Prove that $\triangle ABP \equiv \triangle BCQ$.

## 24*

1. Construct a triangle ABC, given that $\angle A = 44°$, $\angle B = 56°$, and that the perimeter of $\triangle ABC$ is 4 in. Measure AB.

2. D is any point on the bisector of $\angle BAC$; DP, DQ are drawn parallel to AB, AC to meet AC, AB respectively at P, Q. Prove that $DP = DQ$.

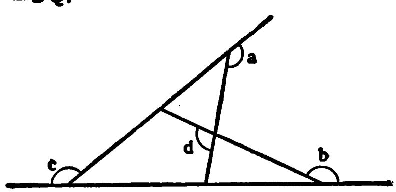


FIG. 455

3. In fig. 455, prove that the sum of the angles $a$, $b$, $c$, $d$ is 6 right angles.

4. ABCD is a parallelogram; ABHK is a square on the same side of AB as CD; BCPQ is a square on the same side of BC as AD. Prove that (i) $\angle QBH = \angle BAD$; (ii) $QH = BD$.

## REVISION PAPERS 25–34 (Theorems 1–33)

[*Including inequalities, intercepts, loci.*]

### 25

**1.** In $\triangle ABC$, $AB = AC$, and $AB$ is produced to $D$. Prove that $\angle ACD - \angle ADC = 2\angle BCD$.

**2.** Given a triangle $ABC$ in which $\angle A > \angle B$, construct *in the simplest way* a point $P$ on $BC$ such that $AP + PC = BC$. State shortly your method.

**3.** In $\triangle PQR$, $PQ = PR$; $S$ is a point on $QR$ produced; $\angle Q = 73°$, $\angle RPS = 38°$. Which is the greater, (i) $RS$ or $RP$, (ii) $SQ$ or $SP$? Give reasons.

**4.** $X$, $Y$ are the mid-points of the sides $AB$, $AC$ of $\triangle ABC$. $BY$ is joined and produced to $Q$ so that $BY = YQ$; $CX$ is joined and produced to $P$ so that $CX = XP$. Prove that (i) $AP = AQ$, (ii) $PAQ$ is a straight line.

### 26

**1.** $PQRS$ is a parallelogram in which $PQ$ is less than $QR$. The bisector $PX$ of $\angle P$ cuts $QR$ at $X$; the bisector $QY$ of $\angle Q$ cuts $PS$ at $Y$. Prove that $PQXY$ is a rhombus.

**2.** Draw a quadrilateral $ABCD$. Construct a point $P$ equidistant from the lines $AB$, $AD$ and such that $PA = PC$. Is there more than one position of $P$? State shortly your method.

**3.** $ABCD$ is a parallelogram. If $\angle BAC > \angle DAC$, prove that $BC > CD$.

**4.** $P$ is a point on the side $AB$ of $\triangle ABC$ such that $AP = \frac{1}{4}AB$; $PQ$ is drawn parallel to $BC$ to meet $AC$ at $Q$. By drawing sets of parallel lines, prove that $PQ = \frac{1}{4}BC$.

### 27

**1.** In $\triangle ABC$, $AB = AC$; $BA$ is produced to $E$. If the bisector of $\angle ACB$ meets $AB$ at $D$, prove that $\angle CDE = \frac{3}{4}\angle CAE$.

**2.** In $\triangle ABC$, $D$ is the mid-point of $BC$. Draw $\triangle ABC$, given that $AB = 5$ cm., $AC = 6$ cm., $AD = 4$ cm. Measure $BC$. [In your sketch, produce $AD$ to $K$ so that $AD = DK$; join $CK$.]

3. In fig. 456, $AB = AC = CP$ and $BQ = BP$. Which is the greater, $QA$ or $QP$? Give reasons.

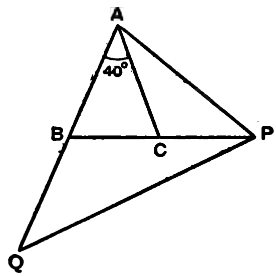



FIG. 456

4. $AB$, $DC$ are the parallel sides of the trapezium $ABCD$; $P$, $Q$ are the mid-points of $AD$, $BC$. Prove that (i) $PQ$ is parallel to $AB$, (ii) $PQ = \frac{1}{2}(AB + DC)$. [Join $P$ and $Q$ to the mid-point $X$ of $BD$.]

## 28

1. In $\triangle ABC$, $AB = AC$; $P$ is any point on $BC$ produced; $PX$, $PY$ are the perpendiculars from $P$ to $AB$, $AC$ produced. Prove that $\angle BPX = \angle BPY$.

2. Draw a triangle $ABC$ in which $AB = 2$ in., $AC = 4$ in., $\angle A = 60°$. Construct two points $P$, $Q$ such that $BP = BQ = 1{\cdot}5$ in., $PA = PC$, and $QA = QC$. State shortly your method.

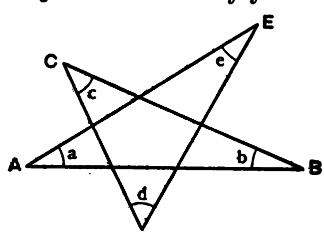



FIG. 457

3. In fig. 457, prove that the sum of the angles $a$, $b$, $c$, $d$, $e$ is two right angles. [Join $CE$.]

4. $ABCD$ is a quadrilateral in which $AB = AD$ and $\angle DBC$ is a right angle. Prove that the line through $A$ parallel to $BC$ bisects $CD$.

## 29

1. ABCD is a square; AB and BC are produced to P and Q respectively so that $BP = CQ$. Prove that PD and AQ are equal and cut at right angles.

2. In fig. 458, PQN is the perpendicular bisector of AB. If BQR bisects $\angle AQP$, find $\angle QAB$ and prove that $QN = \frac{1}{2}QA$.

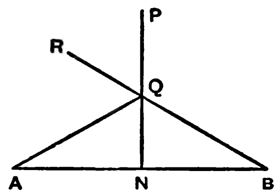


FIG. 458

3. AD, BC are the parallel sides of the trapezium ABCD; AP is the perpendicular from A to the bisector of $\angle ABC$. Prove that AP bisects $\angle BAD$. If PN is the perpendicular from P to BC, what can you say about the circle, centre P, radius PN?

4. In the convex quadrilateral ABCD, the sides AB and CD are equal but not parallel. P, Q, R, S are the mid-points of AD, AC, BD, BC, respectively. Prove that PQRS is a rhombus.

## 30

1. Draw a triangle ABC in which $AB = 5{\cdot}5$ cm., $BC = 4{\cdot}5$ cm., $CA = 4$ cm. Construct a point P equidistant from the lines AB, AC and at a distance of 2 cm. from BC. Is there more than one position of P? State shortly your method.

2. In $\triangle ABC$, $AB = AC$; the bisector of $\angle ABC$ meets AC at D; P is a point on AC produced such that $\angle ABP = \angle ADB$. Prove that (i) $\angle ABD = \angle APB$, (ii) $CB = CP$.

3. The side BC of the equilateral triangle ABC is produced to D so that $BC = CD$; P is any point on BA produced. Prove that (i) $\angle BAD$ is a right angle, (ii) $PD + DC > PB$.

4. Given two points A, B on opposite sides of a given line CD, construct a point P on CD such that $\angle APC = \angle BPC$. State your method and prove that it is correct. (See p. 176, No. 25.)

## 31

1. In $\triangle$ ABC, AB = AC. From a point P on AB a line is drawn perpendicular to BC and meets CA produced at Q. Prove that AP = AQ.

2. In $\triangle$ ABC, AD is the perpendicular from A to BC, and X is the mid-point of BC. Given that AD = 3 cm., AX = 4 cm., BC = 6 cm., draw $\triangle$ ABC. Explain shortly your method.

3. The side BC of the equilateral triangle ABC is produced to Q. Arrange in order of length, the shortest first, the sides of $\triangle$ ABQ. Give reasons.

4. Fig. 459 represents any tetrahedron, *i.e.* a pyramid on a triangular base. P, Q, R, S are the mid-points of the edges AB, AC, DB, DC. Prove that PQSR is a parallelogram.

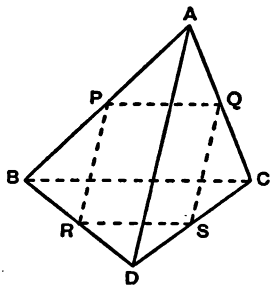


FIG. 459

## 32

1. In fig. 460, AB = AC and CP = CQ. Prove that $\angle$ SRP = 3$\angle$ RPC.

2. In $\triangle$ ABC, D is the mid-point of BC. Given that AD = 3 cm., $\angle$BAD = 30°, $\angle$DAC = 50°, draw the triangle ABC. State shortly your method.

3. ABCD is a parallelogram; ABP, BCQ, CDR, DAS are equilateral triangles outside ABCD. Prove that PQRS is a parallelogram. [Use the theorem: a quadrilateral is a parallelogram if its opposite sides are equal.]

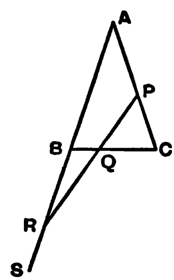


FIG. 460

4. If, in fig. 460, AB = AC and PQ = QR, prove that CP = BR. [Draw PK parallel to BR to meet BC at K.]

## 33*

1. A, B, C are three points on a minor arc of a circle, centre O (*i.e.* an arc less than a semicircle). Prove that $\angle$ ABC is equal to the sum or difference of $\angle$ OAB, $\angle$ OCB. [Two cases: (i) for arc ABC, (ii) for arc ACB.]

2. In fig. 461, P, Q, R are the mid-points of the sides AB, BC, CD of the quadrilateral ABCD and also the mid-points of the sides A′B′, B′C′, C′D′ of the quadrilateral A′B′C′D′. Prove that S is the mid-point of AD and of A′D′.

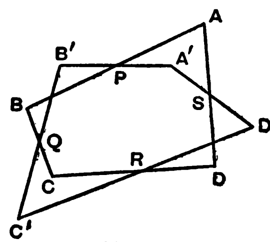



FIG. 461

3. The sides AQ, AR of the rectangle AQPR lie along fixed lines AB, AC. P is a variable point such that PQ − PR = 1 in. Find precisely the locus of P. [Find two fixed lines from which P is equidistant.] How is the locus affected if Q, R may lie on BA produced, CA produced?

4. P is the mid-point of the side AB of the parallelogram ABCD; BD cuts CP at Q and cuts CA at K. Prove that BQ = 2QK.

## 34*

1. In fig. 462, ABCDEFGH is a regular octagon cut out of the square PQRS. Prove that QD = QO. [You may assume that O is the centre of the octagon.]

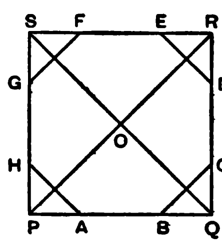



FIG. 462

2. In $\triangle$ ABC, AB = AC and $\angle$ B = 70°. Given that the perimeter of $\triangle$ ABC is 10 cm., draw the triangle and measure AB. [In your sketch draw the straight line PBCQ such that PB = BA = CA = CQ; join AP, AQ.]

3. In $\triangle$ ABC, D is the mid-point of BC, and E is a point on AC such that AE = $\frac{1}{3}$AC; AD cuts BE at K. Prove that KE = $\frac{1}{4}$BE. [Through E draw EP parallel to AD to cut BC at P.]

4. In $\triangle$ BAC, $\angle$ A is a right angle; O is the centre of the square BPQC outside $\triangle$ ABC. Prove that AO bisects $\angle$ BAC. [Prove that O is equidistant from the lines AB, AC.]

# *PART II*

## (*Section* 1)

## AREAS

**Measurement of Area.** Just as the length of a straight line is measured by comparing it with some standard length, such as 1 in., 1 yd., 1 cm., etc., so the **area** of a plane figure—that is, *the amount of plane surface enclosed by the figure*—is measured by comparing it with the area of some standard figure. Any given polygon could be taken as the standard, but the most convenient figure to select as the standard is a **square.**

If each side of a square is 1 in., we say that the area of the square is 1 square inch, *written* 1 sq. in.; if each side of the square is 1 cm., the area of the square is 1 sq. cm., and so on.

**Definition.** A **unit of area** is the area of a square whose side is of unit length.

If one of the small squares of a squared blackboard is taken as the unit of area (its side need not be an exact number of inches or centimetres), and if various figures are drawn on this blackboard, their areas may be expressed in terms of the unit of area by counting up the number of squares they contain. A fair approximation is obtained by counting more than half a square as a whole, and ignoring less than half a square. In this way the areas of the various figures can be compared with one another.

If the lengths of two adjacent sides of a rectangle are 7 units and 4 units, the rectangle can be divided by lines parallel to the sides into $7 \times 4$ compartments, each of which is a square of unit area. Since the rectangle contains 28 squares of unit area, we say that its area is 28 units of area.

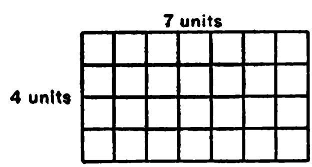


FIG. 463

Consider a rectangle $2\frac{1}{2}$ in. long, $1\frac{1}{3}$ in. wide. This rectangle cannot be divided up into a whole number of squares of side 1 in. But if we choose $\frac{1}{6}$ inch as the unit of length,

length of rectangle = 15 units, width of rectangle = 8 units.

$$\therefore \text{ area of rectangle} = 15 \times 8 \text{ units of area.}$$

But in this case a square of side 1 in. is 6 units long, 6 units wide, and therefore contains $6 \times 6$ units of area,

$$\textit{i.e.}\ 1 \text{ sq. in.} = 6 \times 6 \text{ units of area.}$$

$$\therefore \text{ area of rectangle} = \frac{15 \times 8}{6 \times 6} \text{ sq. in.} = \left(\frac{15}{6} \times \frac{8}{6}\right) \text{ sq. in.}$$
$$= (2\tfrac{1}{2} \times 1\tfrac{1}{3}) \text{ sq. in.}$$

Hence we see that, whether the lengths of the sides of a rectangle are measured by whole numbers or fractions, the area of the rectangle is measured by the product of the measures of its sides.

### Examples for Oral Discussion

1. Find a unit of length for which the sides of the following rectangles are measured by whole numbers. Express the lengths in terms of this unit; state the unit of area and find the area of each rectangle in sq. in.

(i) $2\frac{1}{2}$ in. long, $1\frac{3}{4}$ in. wide; (ii) 4·2 in. long, 1·5 in. wide;
(iii) $1\frac{3}{8}$ in. long, $2\frac{1}{4}$ in. wide; (iv) $1\frac{5}{6}$ in. long, $1\frac{1}{8}$ in. wide.

2. The scale of the plan of an estate is 4 in. to the mile. What area is represented by

(i) a square of side 1 in. on the plan?
(ii) a square of side 0·8 in. on the plan?
(iii) a rectangle whose area is 3·2 sq. in.?

What area on the map represents $\frac{1}{2}$ sq. mile?

## THEOREM 34

**The area of a rectangle is measured by the product of the measures of two adjacent sides.**

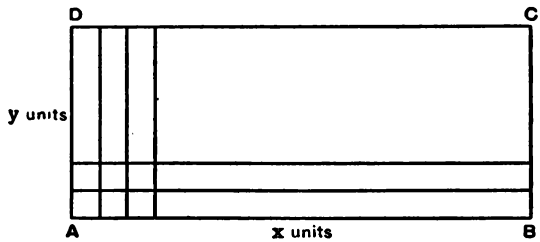



FIG. 464

**Given** a rectangle ABCD in which AB is $x$ units long, AD is $y$ units long, and the unit is chosen so that $x$ and $y$ are integers.

**To prove that** the area of ABCD is $xy$ units of area.

**Construction.** Divide AB into $x$ equal parts and through each point of division draw a line parallel to AD to meet DC. Divide AD into $y$ equal parts and through each point of division draw a line parallel to AB to meet BC.

**Proof.** The lines parallel to AD divide the rectangle ABCD into $x$ equal rectangles, the lengths of whose adjacent sides are 1 unit and $y$ units.
The lines parallel to AB divide each of these rectangles into $y$ equal squares, the length of each side being 1 unit.

$\therefore$ ABCD contains $xy$ squares, each of unit area;

$\therefore$ the area of ABCD is $xy$ units of area.

NOTE. This proof fails if the lengths of AB and AD cannot be measured by integers or fractions. In this case they are called *incommensurable*.

**Notation.** The rectangle ABCD is often called rect. AC or rect. BD, if this does not cause any ambiguity.

The rectangle ABCD is said to be *contained* by AB, AD or by AB, BC, *i.e.* by any pair of adjacent sides.

*For brevity*, we say that

$$\text{area of rect. } ABCD = AB \times AD \textit{ or } AB.AD,$$

but *it must be clearly understood that this is merely an abbreviation* for the statement that the number of units of area of the rectangle ABCD is the product of the numbers of units of length of AB and AD.

If ABCD is a square, it is often spoken of as *the square on* AB or more shortly, *sq. on* AB.

*For brevity*, we say that

$$\text{area of sq. on } AB = AB^2,$$

but *this is merely an abbreviation* for the statement that the number of units of area of the square ABCD is the square of the number of units of length of AB.

If two figures are equal in area, they are called **equivalent**.

The symbol for "*is equivalent to*" is $=$. The statement $\triangle ABC = \triangle XYZ$, means that the triangles ABC, XYZ are equal in area; it does *not* mean that they are congruent.

### Examples for Oral Discussion

1. Find the area of the right-angled triangle ABC, given that

$$AB = 7 \text{ in.}, \quad BC = 3 \text{ in.}, \quad \angle ABC = 90°.$$

Complete the parallelogram ABCD.

(i) Explain why ABCD is a rectangle.
(ii) What is the area of rect. DABC?
(iii) What is the area of $\triangle ABC$? Give reasons.

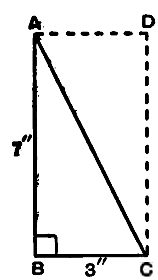



Fig. 465

2. What expression (using capital letters) represents the area of $\triangle PQR$ if $\angle QPR$ is a right angle?

## EXERCISE 49

*[The unit of length in each diagram is 1 in.]*

**1.** Draw on squared paper a rectangle, 2·3 in. long, 1·4 in. wide. Calculate its area and verify by counting squares.

**2.** Sketch a figure to illustrate that 1 sq. yd. = 9 sq. ft.

Find the areas of figs. 466–468 in which all the corners are right-angled:

**3.**

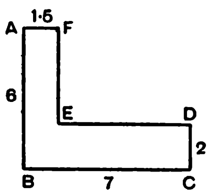



FIG. 466

**[4]**

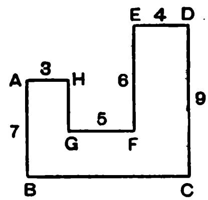



FIG. 467

**5.**

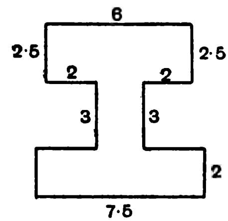



FIG. 468

**6.** A rectangle is equal in area to a square, side 4 in. Find its breadth if its length is 10 in.

**[7]** Find the length of the side of a square which is equal in area to a rectangle 4 ft. long, 3 in. wide.

**8.** Copy fig. 469 and insert in each rectangular compartment its area. Complete $(a+b)^2 = \ldots$

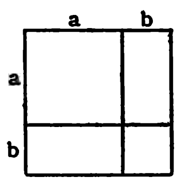



FIG. 469

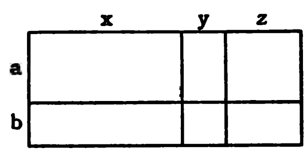



FIG. 470

**[9]** Copy fig. 470 and insert in each rectangular compartment its area. Complete $(a+b)(x+y+z) = \ldots$

**10.** Draw a figure like fig. 469 to illustrate that $(2x)^2 = 4x^2$. Complete the statement: If a straight line AB is bisected at C, then $AC^2 = \ldots AB^2$.

**[11]** Draw a figure to illustrate that $(3y)^2 = 9y^2$. State the corresponding geometrical theorem.

12. A map is drawn on a scale of 4 miles to the inch.
   (i) What area is represented by $\frac{1}{2}$ sq. in. on the map?
   (ii) What area on the map represents 4 sq. miles?

[13] A map is drawn on a scale of $\frac{1}{5}$ mile to the inch.
   (i) What area is represented by 15 sq. in. on the map?
   (ii) What area on the map represents 0·8 sq. mile?

[14] On a map in which 6 in. represents a mile, a field is a square measuring $\frac{1}{2}$ in. each way. Find the area of the field in acres, correct to $\frac{1}{10}$ acre.

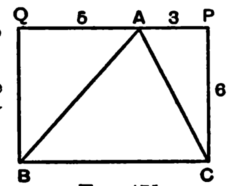


FIG. 471

Make a sketch of fig. 466 and find the areas of the following triangles obtained by joining the necessary points:—

15. (i) △ ABC; (ii) △ AEF.

[16] (i) △ BCD; (ii) △ DEF.

17. In fig. 471, BCPQ is a rectangle. Find the area of △ ABC.

18. In fig. 472, find the area of the quadrilateral ABCD.

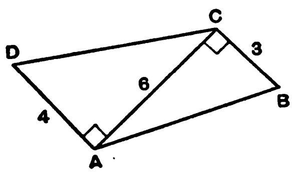


FIG. 472

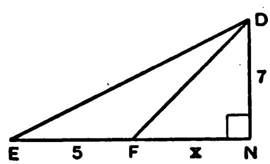


FIG. 473

19. In fig. 473, EFN is a straight line. Find the area of △ DEF.

20. In fig. 474, find the area of
   (i) △ BCQ; (ii) quad. APBC;
   (iii) △ ABC.

[21] B is 100 yd. east and 30 yd. north of A; C is 40 yd. east and 90 yd. north of A. Find the area of △ ABC. [Draw a figure like fig. 474.]

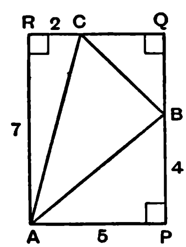


FIG. 474

*Use squared paper for Nos.* 22–26:

22. Find the area of the triangle whose vertices are the points (2, 1); (2, 5); (4, 7).

23. Repeat No. 22 for the points (3, 2); (5, 4); (4, 8).

[24] Repeat No. 22 for the points (1, 1); (5, 2); (6, 5).

**25.** Find the area of the quadrilateral whose vertices are the points (0, 0); (3, 2); (1, 5); (0, 7).

[26] Repeat No. 25 for the points (1, 3); (3, 2); (5, 5); (2, 7).

**27.** In $\triangle$ ABC, $\angle$ A $=90°$ and AB $=6$ cm. If the area of $\triangle$ ABC is 15 sq. cm., find the length of AC.

[28] AD is the perpendicular from A to the side BC of the acute-angled triangle ABC. If BD $=6$ in. and DC $=2$ in., and if the area of $\triangle$ ABC is 12 sq. in., find the length of AD.

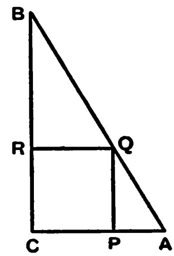


FIG. 475

**29.** In fig. 475, CPQR is a square and AQB is a straight line; CA $=6$ cm., CB $=10$ cm. If CP $=x$ cm., write down in terms of $x$ the areas of $\triangle$ CAQ, $\triangle$ CBQ. What is the area of $\triangle$ ABC? Find the value of $x$.

***30.** ABCD is a square, side 4 in.; AB is produced to P, and AD is produced to Q so that AP $=12$ in., AQ $=6$ in. Calculate the areas of $\triangle$ ACP, $\triangle$ ACQ, $\triangle$ APQ and then *prove* that PCQ is a straight line.

***31.** If in fig. 475, CPQR is a rectangle and AQB is a straight line, and if CP $=x$ in., CR $=y$ in., CA $=a$ in., CB $=b$ in., find an equation connecting $x$, $y$, $a$, $b$.

***32.** Fig 476, not drawn to scale, in which all the corners are right-angled, represents the plan of an estate of area 5 acres. The given measurements are in inches. On what scale, inches to the mile, is the plan drawn? A line perpendicular to AB cuts AB at Q and is drawn so that it bisects the area of the estate. Find the length of AQ on the plan.

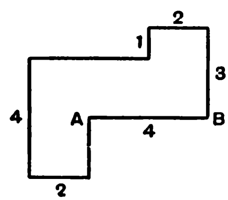


FIG. 476

***33.** Draw on squared paper a circle of radius 2 in. and find approximately its area by counting small squares. Find the ratio of the area of the circle to the area of the square on the radius.

***34.** Find the area of a pentagon whose vertices are the points (0, 0); (4, 0); (9, 3); (9, 5); (3, 7).

## Distance between Two Parallel Straight Lines

The following facts are easy to prove:—

(1) *If* AB *and* CD *are parallel, and if from any point* P *on* AB *the perpendicular* PX *is drawn to* CD, *then* PX *is also perpendicular to* AB.

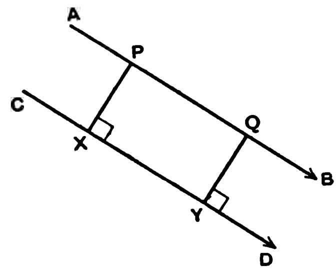


FIG. 477

**Proof:** $\angle APX = \angle PXD$, alt. $\angle$s, AB $\parallel$ CD.

But $\angle PXD = 1$ rt. $\angle$, *given*,

$\therefore$ $\angle APX$ is a right angle.

(2) *If* AB *and* CD *are parallel, and if* PX, QY *are the perpendiculars from points* P, Q *on* AB *to* CD, *then* PX = QY.

**Proof:** Since $\angle PXD = \angle QYD$ *rt.* $\angle$*s, constr.*

PX is parallel to QY.

Also PQ is parallel to XY *given.*

$\therefore$ PQYX is a parallelogram,

$\therefore$ PX = QY *opp. sides* $\parallel$*gram.*

These facts may be expressed as follows:—

**If two straight lines AB, CD are parallel, the length of the perpendicular from any point P on EITHER line to the other line is the same for all positions of P.**

The length of this perpendicular is called **the distance between the parallel lines.**

The statement is often expressed by the words:

**Parallel lines are everywhere the same distance apart.**

But it must be realised that a proof is necessary.

**Altitude or Height.** If any side of a parallelogram is taken as its **base,** the distance between that side and the parallel side is called an **altitude** or **height** of the parallelogram.

Thus, in fig. 478, if AB is taken as the base of the parallelogram ABCD, the altitude is the length of any one of the equal perpendiculars DH, PQ, P′Q′. And in fig. 479, if BC is taken as the base of the same parallelogram ABCD, the altitude is the length of any one of the equal perpendiculars AK, RS, R′S′.

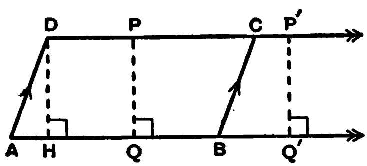



FIG. 478

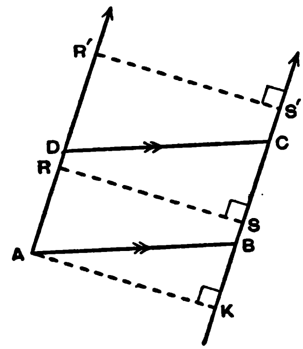



FIG. 479

In general, a parallelogram has two *distinct* bases, *i.e.* bases of different lengths, and two distinct altitudes.

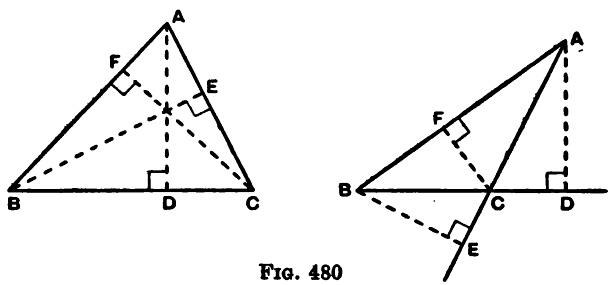



FIG. 480

If any side of a triangle is taken as its base, the perpendicular to that side, produced if necessary, from the opposite vertex, is called an altitude or height of the triangle. Thus, in fig. 480, the perpendiculars AD, BE, CF to BC, CA, AB represent the altitudes of the triangle ABC corresponding to the bases BC, CA, AB respectively.

In general, a triangle has three *distinct* bases, *i.e.* bases of different lengths, and three distinct altitudes.

## Equal Altitudes

If PX, QY are two parallel straight lines, and if

(i) parallelograms are drawn with bases along PX and their opposite sides along QY,
(ii) triangles are drawn with bases along PX and opposite vertices on QY,

then the parallelograms and triangles are all said to be **between the same parallels,** PX and QY.

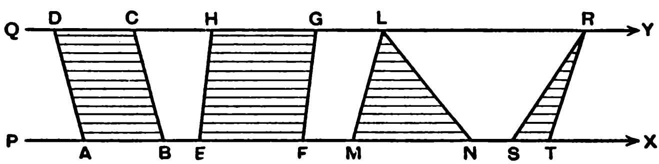



FIG. 481

In fig. 481, if AB, EF are taken as the bases of the parallelograms ABCD, EFGH, and if MN, ST are taken as the bases of the triangles LMN, RST, the corresponding altitudes are all equal because each is equal to the distance between the parallel lines PX, QY. Thus:

**Parallelograms and triangles between the same parallels have equal altitudes.**

Conversely, if we are given parallelograms and triangles of equal altitudes, and if we draw two parallel lines PX, QY whose distance apart is equal to that of the altitudes, we can construct parallelograms and triangles, *between the parallels* PX, QY, congruent to the given parallelograms and triangles.

## Examples for Oral Discussion

1. From a rectangular sheet of paper ABCD cut off a right-angled triangle BNC, see fig. 482. Turn $\triangle$BNC round through 2 right angles and fit it on to the figure ANCD, first with BN along B′C in the position NB′C, and then with BN along DQ in the position ADQ. What fact does this illustrate?

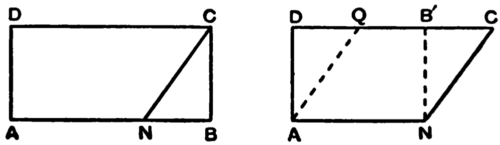



FIG. 482

2. Take a packet of sheets of paper. First hold it so that the front face is a rectangle. Then slide the sheets sideways so that the front face is a parallelogram. Is the height of the packet altered? What fact does this illustrate?

**3. Fig. 483 represents two parallelograms ABCD, ABPQ on the same base AB and between the same parallels AB, DCQP. Prove that area of ABCD = area of ABPQ.**

(i) Use the SAA test to prove that $\triangle$DAQ $\equiv$ $\triangle$CBP.

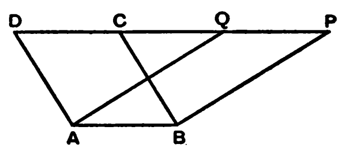



FIG. 483

(ii) What remains if $\triangle$DAQ is taken away from the whole figure DABP and if $\triangle$CBP is taken away from DABP? Complete the proof.

(iii) Draw your own figure, and draw in it a rectangle on the same base AB and between the same parallels as the parallelogram ABCD. How does this help you to calculate the area of a given parallelogram?

4. In fig. 484, △BXC = △BYC. What result is obtained by taking each triangle in turn from the whole figure?

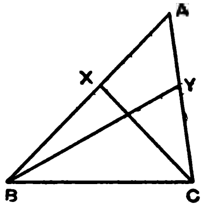


FIG. 484

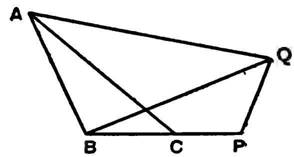


FIG. 485

5. In fig. 485, △ABC = △BPQ. What can you say about the area of the quadrilateral ACPQ?

6. Fig. 486 represents a triangle ABC and a rectangle PQBC on the same base BC and between the same parallels BC, QPA. **Prove that area of △ABC = $\frac{1}{2}$ area of rect. PQBC.**

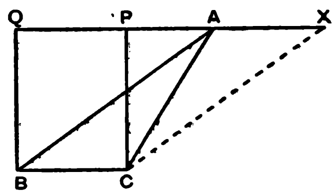


FIG. 486

Complete the parallelogram ABCX.

(i) What do you know about the area of △ABC? Give reasons.

(ii) Use Example 3 to complete the proof.

(iii) How does this help you to calculate the area of a given triangle?

(iv) If in fig. 486, BC = 3 in., BQ = 5 in., find the area of the parallelogram BCXA and the area of △ABC.

Example 3 shows that the area of a parallelogram is equal to that of a rectangle on the same base and between the same parallels, and can be calculated by constructing this rectangle.

Example 6 shows that the **area of a triangle** is half that of a rectangle on the same base and between the same parallels.

7. In fig. 487, AKHQP is a straight line parallel to BC.

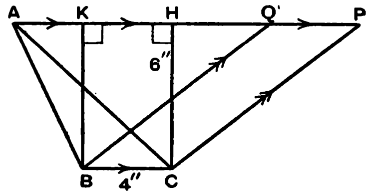



FIG. 487

(i) Find the area of ||gram BCPQ.

(ii) Find the area of △ABC.

(iii) If CP = 8″, find the length of the perpendicular from C to BQ.

(iv) If AC = 7·5″, find the length of the perpendicular from B to AC.

8. Draw a parallelogram ABCD in which AB = 8 cm., AD = 6 cm., ∠BAD = 65°. By making the necessary constructions and measurements, find the area of ABCD in *two* distinct ways.

9. Draw a triangle whose sides are 5 cm., 6 cm., 8 cm. By making the necessary constructions and measurements, find the area of the triangle in *three* distinct ways.

## Area of Parallelogram

If ABCD is a **parallelogram** in which AB = $x$ units and the distance between the parallel sides AB, CD is $h$ units,

**area of parallelogram ABCD = xh units of area.**

This result may be expressed trigonometrically:

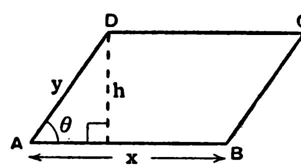



FIG. 488

If AD = $y$ units and if

∠DAB = $\theta$,

$$\frac{h}{y} = \sin\theta \quad \text{or} \quad h = y \sin\theta,$$

∴ **area of parallelogram ABCD = xy sin θ units of area.**

## Area of Triangle

If ABC is a triangle in which BC $=a$ units and the length of the perpendicular from A to BC is $h$ units,

**area of triangle ABC $=\frac{1}{2}ah$ units of area.**

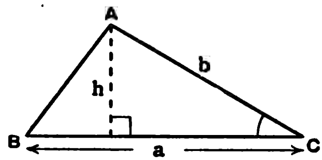



FIG. 489

The result may be expressed trigonometrically:

If AC $=b$ units and if $\angle$ACB is acute and is denoted by C,

$$\frac{h}{b}=\sin \text{C} \quad \text{or} \quad h=b\sin \text{C},$$

**$\therefore$ area of triangle ABC $=\frac{1}{2}ab\sin$ C units of area.**

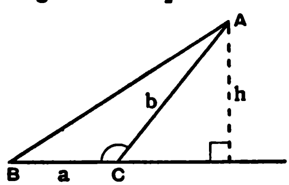



FIG. 490

If $\angle$ACB is obtuse, see fig. 490, the same argument shows that the area is

$$\tfrac{1}{2}ab\sin(180° - \text{C}) \text{ units of area.}$$

On pp. 75, 79, definitions were given only for the sines, cosines, tangents of *acute* angles. It is *convenient* to define the sine of an obtuse angle as equal to the sine of its supplement,

*e.g.* by definition, $\sin 150° = \sin 30°$, $\sin 110° = \sin 70°$, etc.

Therefore by definition if $\angle$ACB is obtuse,

$$\sin \text{C} = \sin(180° - \text{C}).$$

Hence, whether $\angle$ACB is acute or obtuse,

**area of triangle ABC $=\frac{1}{2}ab\sin$ C units of area.**

## Area of Trapezium

ABCD is a trapezium in which AB is parallel to DC. If $AB = a$ units, $DC = b$ units, and if the distance between the parallel lines AB, DC is $h$ units,

**area of trapezium ABCD $= \frac{1}{2}h(a+b)$ units of area.**

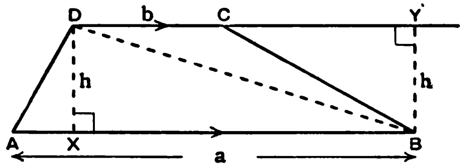



FIG. 491

Join BD and draw the altitudes DX, BY of $\triangle$DAB, $\triangle$BDC.

area of $\triangle DAB = \frac{1}{2} DX . AB = \frac{1}{2} h a$ units of area,

area of $\triangle BDC = \frac{1}{2} BY . DC = \frac{1}{2} h b$ units of area,

$\therefore$ area of trapezium $ABCD = \frac{1}{2} ha + \frac{1}{2} hb$ units of area,

$= \frac{1}{2} h(a+b)$ units of area.

This result may be stated as follows:—

**The area of a trapezium is measured by the product of the measures of half the sum of the parallel sides and the distance between them,**

*or*, more shortly, **height $\times$ average width.**

## Area of Quadrilateral

The area of any quadrilateral ABCD can be found by joining BD, and finding the areas of each of the triangles ABD, CBD. But it is better to construct a triangle whose area is equal to that of the given quadrilateral. The method is given on p. 263.

### Example for Oral Discussion

If the lengths of the sides of a triangle are $a$, $b$, $c$ units, and if $s=\frac{1}{2}(a+b+c)$, *i.e.* if the semi-perimeter is $s$ units, it can be proved that the area of the triangle is

$$\sqrt{\{s(s-a)(s-b)(s-c)\}} \text{ units of area.}$$

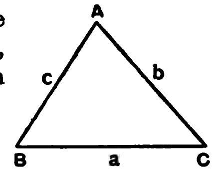


FIG. 492

Find the area of a triangle whose sides are 15 in., 14 in., 13 in., and calculate the length of each altitude.

Draw your own figure and mark the data on it. Copy and complete the following:—

| | | |
|---|---|---|
| $a=15$ | $s-a=\ldots$ | Why is the sum of $s-a$, $s-b$, $s-c$ equal to $s$? This provides a useful check. |
| $b=14$ | $s-b=\ldots$ | |
| $c=13$ | $s-c=\ldots$ | |
| $a+b+c=\ldots$ | *Add* $\ldots=\ldots$ | |
| $\therefore s=\ldots$ | | |

$\therefore$ area of $\triangle$ ABC $=\sqrt{(\ \ldots\ )}$ sq. in. $=\ \ldots$ sq. in.

If the perpendicular from A to BC is $p$ in.,

$$\text{area of } \triangle \text{ABC} = \tfrac{1}{2}\times 15\times p \text{ sq. in.}$$

Hence find $p$.

Find in a similar way the other two altitudes.

## NUMERICAL EXAMPLES

### EXERCISE 50

[*The unit of length in each diagram is* 1 *in.*]

**1.** Draw a parallelogram ABCD in which AB $=5$ cm., AD $=4$ cm., $\angle$BAD $=65°$. Construct two rectangles equal in area to it, one with AB as base, the other with AD as base. Measure the heights of the two rectangles and then find the area of ABCD in *two* ways.

If you know the trigonometrical formula for the area of a parallelogram, find the area of ABCD in a third way.

2. In fig. 493, AP, AQ are altitudes of the parallelogram ABCD.

(i) If AQ = 4 cm., CD = 5 cm., find the area of ABCD.
(ii) If area of ABCD = 24 sq. in., AB = 6 in., find AQ.
(iii) If AB = 5 in., AP = 4 in., AD = 6 in., find AQ.

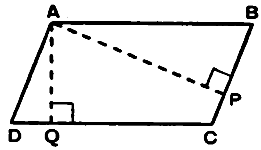


FIG. 493

3. In fig. 494, AD, BE, CF are altitudes of △ ABC.

(i) If AD = 7 in., BC = 5 in., find area of △ ABC.
(ii) If area of △ ABC = 40 sq. cm., AC = 8 cm., find BE.
(iii) If BE = 5 in., AB = 6 in., CF = 4 in., find AC.

[4] ABCDE is a straight line such that AB = 6 cm., BC = 2 cm., CD = 4 cm., DE = 2 cm.; P is a point 5 cm. from the line AE.

(i) Find the areas of △ PAB, △ PBD, △ PDE.
(ii) What fraction is △ PCD of △ PAE?
(iii) What triangle is equal in area to △ PBE?

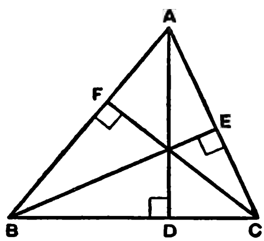


FIG. 494

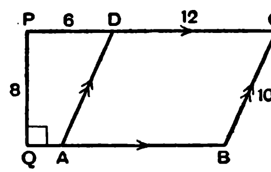


FIG. 495

5. In fig. 495, find (i) the area of ABCD, (ii) the length of the perpendicular from B to AD.

[6] Sketch fig. 495 and join AP, QD, QC. Find (i) the area of △ APD, (ii) the area of △ QDC, (iii) the length of the perpendicular from P to AD.

7. In fig. 496, find (i) the area of △ ABC, (ii) the length of the perpendicular from B to AC.

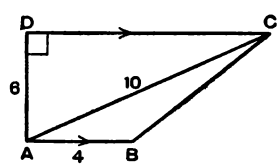


FIG. 496

8. In fig. 497, find

(i) the areas of ABCD, $\triangle$ PBQ, $\triangle$ BCP;

(ii) the lengths of the perpendiculars from B to AD, from Q to BP, from C to PB;

(iii) the area of AQPD.

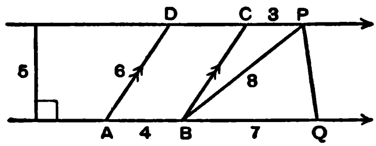



FIG. 497

9. The distance between the parallel sides AB, DC of a trapezium ABCD is 4 in. If AB = 8 in., DC = 5 in., find the area of ABCD.

[10] Make a sketch of fig. 467, p. 237.

(i) Join BG, CF and find the area of BCFG.

(ii) Join CG, HD and find the area of CGHD.

11. ABXY is a parallelogram of area 18 sq. cm.; AB = 6 cm., AY = 4 cm. and C is a point YX produced such that BC = 5 cm. Find (i) area of $\triangle$ ABC; (ii) the length of the perpendicular from B to AY; (iii) the length of the perpendicular from A to CB, produced if necessary.

12. ABCD is a parallelogram of area 24 sq. cm.; AB = 6 cm., AD = 5 cm. Calculate the lengths of the corresponding altitudes. Draw the parallelogram and measure one of its acute angles. If you know the trigonometrical formula for the area of a parallelogram, find also this angle by calculation.

[13] PQRS is a parallelogram of area 18 sq. cm.; PQ = 5 cm., QR = 4 cm. Calculate the lengths of the corresponding altitudes.

[14] ABCD is a parallelogram in which AB = 3 in., BC = 12 in., and the perpendicular from B to AD is 2·5 in. Find the length of the perpendicular from A to CD.

[15] BE and CF are two altitudes of $\triangle$ ABC. If AB = 6 in., BE = 5 in., CF = 4 in., find the length of AC.

16. P is a point on the base BC of $\triangle$ ABC such that BP = 3 in., PC = 2 in. If the area of $\triangle$ APC is 5 sq. in., find the area of $\triangle$ APB.

[17] A triangle and a rectangle have the same base and the height of the rectangle is 6 in. If the area of the triangle is $\frac{1}{3}$ of that of the rectangle, find the height of the triangle.

18. Find the area of a rhombus whose diagonals are 5 in., 6 in. long.

[19] The area of a rhombus is 25 sq. cm. and one diagonal is twice as long as the other. Find the length of each diagonal.

20. ABCD is a trapezium in which AB, DC are parallel. If AB = 7 in., DC = 3 in., and if the area of ABCD is 18 sq. in., find the area of $\triangle$ BCD.

21. The lengths of the sides of a triangle are 13 in., 20 in., 21 in. Use the formula on p. 248 to calculate its area and find the length of the shortest altitude.

[22] Repeat No. 21 for a triangle whose sides are of lengths 10 in., 17 in., 21 in.

*23. In $\triangle$ ABC, AB = 8 cm., AC = 9 cm., and D is the mid-point of BC. If the area of $\triangle$ ABC is 36 sq. cm., find the lengths of the perpendiculars from D to AB and AC.

*24. The lengths of the two altitudes of a parallelogram are 2 in., 3 in., and the perimeter of the parallelogram is 20 in. Find the lengths of the sides of the parallelogram.

*25. The lengths of the three altitudes of a triangle are $6x$, $4x$, $3x$ in., and the perimeter of the triangle is 18 in. Find the lengths of the sides of the triangle.

*26. D, E are points on the sides AC, AB respectively of $\triangle$ ABC such that AD = $\frac{1}{3}$AC and AE = $\frac{2}{3}$AB. If the area of $\triangle$ ABC is 18 sq. cm., find the areas of (i) $\triangle$ BCE, (ii) $\triangle$ CDE, (iii) $\triangle$ ADE.

*27. P, Q, R are points on the sides BC, CA, AB respectively of $\triangle$ ABC such that BP = 6 in., PC = 8 in., CQ = 6 in., QA = 9 in., AR = 10 in., RB = 3 in. Prove that the area of $\triangle$ ABC is 84 sq. in. and find the areas of (i) $\triangle$ APC, (ii) $\triangle$ QPC, (iii) $\triangle$ RPB, (iv) $\triangle$ RQA, (v) $\triangle$ PQR.

*28. **The Sine Formula.** If in any triangle ABC, BC = $a$ units, CA = $b$ units, AB = $c$ units, prove that $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$. (See p. 246.)

## THEOREM 35

**The area of a parallelogram is equal to the area of a rectangle on the same base and between the same parallels.**

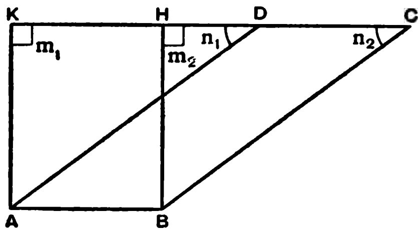



FIG. 498

**Given** a parallelogram ABCD and a rectangle ABHK on the same base AB and between the same parallels AB, KHDC.

**To prove that** area of ABCD = area of ABHK.

**Proof.** With the notation in the figure in $\Delta$s AKD, BHC,

$m_1 = m_2$ *corr.* $\angle$*s*, AK ∥ BH.
$n_1 = n_2$ *corr.* $\angle$*s*, AD ∥ BC.
AK = BH *opp. sides* ∥gram.

$\therefore$ $\Delta$s $\begin{matrix}\text{AKD}\\\text{BHC}\end{matrix}$ are congruent AAS.

$\therefore$ area of $\Delta$AKD = area of $\Delta$BHC.

**From the whole figure ABCK, subtract each triangle in turn,**
then the remainders ABCD, ABHK are equal in area.

NOTE. In order to obtain a proof which holds for all figures, *i.e.* whether D lies between K and H or on KH produced, and which holds if ABHK is any parallelogram, it is necessary

(i) to use the AAS test for proving the triangles congruent,
(ii) to *subtract* the areas of these triangles in turn *from the whole* figure.

**Corollary 1. Parallelograms on the same base and between the same parallels are equal in area.**

The proof of Theorem 35 applies, word for word, to any two parallelograms ABCD, ABHK on the same base AB and between the same parallels AB, KHDC.

**Corollary 2. The area of a parallelogram is measured by the product of the measures of its base and corresponding altitude.**

In fig. 498,

area of rectangle ABHK = AB . BH,

$\therefore$ area of parallelogram ABCD = AB . BH.

But BH is the distance between the parallel lines AB, KHDC, and is therefore equal to the altitude of the parallelogram ABCD corresponding to the base AB.

$\therefore$ Area of parallelogram ABCD is measured by the product of the measures of the base AB and the corresponding altitude.

**Corollary 3. Parallelograms on equal bases and between the same parallels are equal in area.**

Since the parallelograms are between the same parallels, their altitudes are equal; also the corresponding bases are given equal.

Therefore their areas are equal.

NOTE. If a proof of any of these corollaries is asked for in an examination, the proof of the original theorem must be given, and when necessary supplemented as shown.

**Important Hint.** In theoretical rider work, it will be found that Corollary 1 is far more useful than Corollary 2 or than Theorem 35 itself.

### THEOREM 36

**The area of a triangle is equal to one-half of the area of a rectangle on the same base and between the same parallels.**

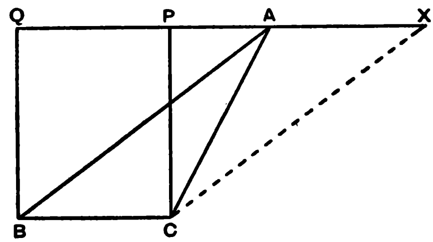



FIG. 499

**Given** a triangle ABC and a rectangle PQBC on the same base BC and between the same parallels BC, QPA.

**To prove that** area of ABC $= \frac{1}{2}$ area of PQBC.

**Construction.** Through C draw CX parallel to BA to meet QA or QA produced at X.

**Proof.** CX is parallel to BA *constr.*,
BC is parallel to AX *given*,

$\therefore$ ABCX is a parallelogram.

$\therefore$ XABC, PQBC are parallelograms on the same base BC and between the same parallels BC, QPAX,

$\therefore$ area XABC = area PQBC.

Since the diagonal AC bisects the area of the parallelogram XABC,

area ABC $= \frac{1}{2}$ area XABC,
$\therefore$ area ABC $= \frac{1}{2}$ area PQBC.

**Corollary 1. The area of a triangle is measured by half the product of the measures of its base and altitude.**

In fig. 499, area of rectangle $PQBC = BC \,.\, CP$,

$\therefore$ area of $\triangle ABC = \frac{1}{2} BC \,.\, CP$.

But the perpendicular from $A$ to $BC$ is equal to $CP$, opposite sides of a rectangle,

$\therefore$ area of $\triangle ABC = \frac{1}{2}$ base $\times$ altitude.

**Corollary 2. Triangles on the same or equal bases and of equal altitudes are equal in area.**

This follows from Corollary 1.

**Corollary 3. If triangles of the same area have the same or equal bases, their altitudes are equal.**

This follows from Corollary 1.

The proof of Theorem 36 may be used *without any alteration* to prove that

**The area of a TRIANGLE is equal to one half of the area of a PARALLELOGRAM on the same base and between the same parallels.**

An alternative proof is given on p. 258.

**Important Hint.** In theoretical rider work, it will be found that Corollary 2 is far more useful than Corollary 1 or than Theorem 36 itself.

## THEOREM 37

**Triangles on the same base and between the same parallels are equal in area.**

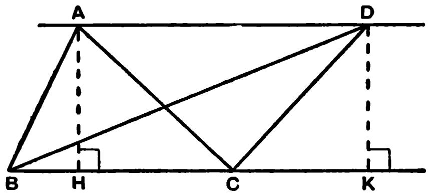



FIG. 500

**Given** two triangles ABC, DBC on the same base BC and between the same parallels BC, AD.

**To prove that** area of ABC = area of DBC.

**Construction.** From A, D draw the perpendiculars AH, DK to BC, produced if necessary.

**Proof.** Area of ABC $= \frac{1}{2}$ BC . AH  $\frac{1}{2}$ *base* × *altitude.*
Area of DBC $= \frac{1}{2}$ BC . DK  $\frac{1}{2}$ *base* × *altitude.*

Since AH and DK are perpendicular to BC,
AH is parallel to DK,

also HK is parallel to AD  *given,*

∴ AHKD is a parallelogram,

∴ AH = DK  *opp. sides* ‖*gram.*

∴ area of ABC = area of DBC.

**Corollary. Triangles on equal bases and between the same parallels are equal in area.**

The proof is similar.

## THEOREM 38

**Triangles of equal area on the same base and on the same side of it are between the same parallels.**

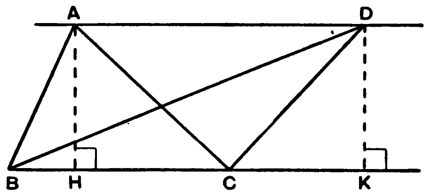



FIG. 501

**Given** two triangles ABC, DBC on the same side of the same base BC and such that $\triangle ABC = \triangle DBC$.

**To prove that** AD is parallel to BC.

**Construction.** From A, D draw the perpendiculars AH, DK to BC, produced if necessary.

**Proof.** Area of $ABC = \frac{1}{2} BC \,.\, AH$ — $\frac{1}{2}$ *base* $\times$ *altitude.*

Area of $DBC = \frac{1}{2} BC \,.\, DK$ — $\frac{1}{2}$ *base* $\times$ *altitude.*

But area of $ABC =$ area of $DBC$ — *given,*

$\therefore\ AH = DK.$

Since AH and DK are perpendicular to BC,

AH is parallel to DK.

$\therefore$ AH and DK are equal and parallel.

$\therefore$ AHKD is a parallelogram.

$\therefore$ AD is parallel to HK or BC.

**Corollary. Triangles of equal area on equal bases in the same straight line and on the same side of it are between the same parallels.**

The proof is similar.

## THEOREM 39

**If a triangle and a parallelogram are on the same base and between the same parallels, the area of the triangle is equal to half that of the parallelogram.**

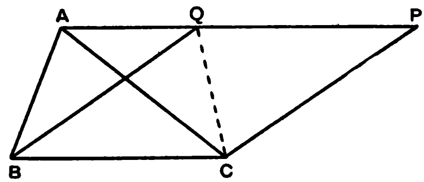



FIG. 502

**Given** a triangle ABC and a parallelogram PQBC on the same base BC and between the same parallels BC, AQP.

**To prove that** area of ABC $= \frac{1}{2}$ area of PQBC.

**Construction.** Join CQ.

**Proof.** The triangles ABC, QBC are on the same base BC and between the same parallels BC, AQ,

$\therefore$ area of ABC = area of QBC.

Since the diagonal CQ bisects the area of the parallelogram PQBC,

area of QBC $= \frac{1}{2}$ area of PQBC.
$\therefore$ area of ABC $= \frac{1}{2}$ area of PQBC.

NOTE. If preferred, this theorem may be proved by using the construction and method of Theorem 37.

*Alternatively*, the proof of Theorem 36 (with the construction) applies, word for word, to this theorem.

**Area Theorems and Rider Work.** Although the properties of areas of triangles have been deduced from those of parallelograms, it will be found that the theorems about the areas of triangles are more useful than those about the areas of parallelograms in rider work.

Most of the riders in the next exercise can be solved by using one or more of the following theorems (proved in the previous pages) with which pupils must be familiar:—

**1. If, in fig. 503, AD is parallel to BC, then △ABC =△DBC.**

**2. If, in fig. 503, △ABC =△DBC, then AD is parallel to BC.**

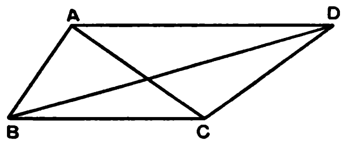


FIG. 503

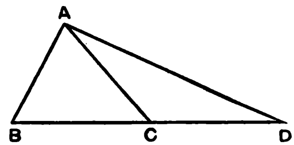


FIG. 504

**3. If, in fig. 504, BCD is a straight line and BC = CD, then △ABC =△ADC.**

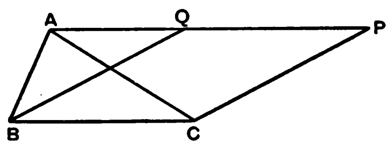


FIG. 505

**4. If, in fig. 505, BCPQ is a parallelogram and if A lies on PQ, or PQ produced, then △ABC = $\frac{1}{2}$ ||gram PQBC.**

## EXERCISE 51

[*Arrows indicate that lines are given parallel.*]

Nos. 1–5 refer to Fig. 506.

1. If AD is parallel to BC, name a triangle equal to △ABC Give reasons.

2. If AD is parallel to BC, prove that △AKB = △DKC.

3. If △KDA = △KCB, prove that DC is parallel to AB.

4. If DK = KB, explain why △DKC = △BKC and prove that △DAC = △BAC.

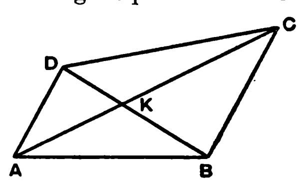


FIG. 506

[5] If △DAK = △DCK, prove that △BAK = △BCK.

**6.** If, in fig. 507, YP = PZ and XQ = QP, prove that

$$\triangle QXY = \triangle QPZ.$$

[7] If, in fig. 507, YP = PZ, prove that $\triangle XYQ = \triangle XZQ$.

[8] K is a point on the side BC of $\triangle ABC$. What can you say about BK if $\triangle ABK = \frac{2}{3}$ of $\triangle ABC$?

**9.** What can you say about the altitudes BE, CF of $\triangle ABC$ if AB = 2AC?

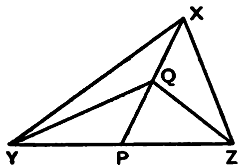



Fig. 507

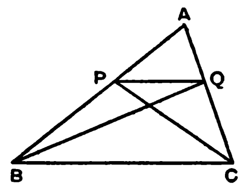



Fig. 508

**10.** If, in fig. 508, PQ is parallel to BC, prove that

$$\triangle APC = \triangle AQB.$$

**11.** If, in fig. 508, AP = PB and AQ = QC, what can you say about $\triangle BPQ$ and about $\triangle CPQ$? Prove that PQ is parallel to BC.

[12] If, in fig. 508, BP = 2PA and CQ = 2QA, what can you say about $\triangle BPQ$ and about $\triangle CPQ$? Prove that PQ is parallel to BC.

[13] If, in fig. 509, AB = EC, prove that $\triangle DAB = \triangle BEC$.

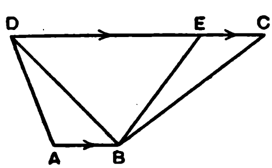



Fig. 509

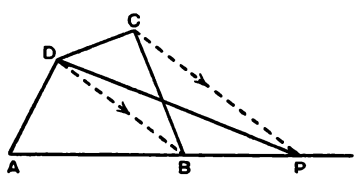



Fig. 510

**14.** In fig. 510, ABP is a straight line, prove that quad. ABCD = $\triangle APD$.

**Use this result to construct a triangle equal in area to any given quadrilateral.**

**15.** If, in fig. 511, DC = FE, prove that the trapeziums ABCD, ABEF are equal, *without* assuming the formula for the area of a trapezium.

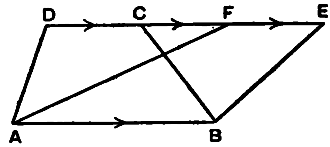



Fig. 511

[16] ABCD is a parallelogram; P is any point on AB between A and B. Prove that

$\triangle APD + \triangle BPC = \triangle DPC$.

**17.** ABCD is a parallelogram; P is any point on AD; Q is any point on AB produced. Prove that $\triangle CPB = \triangle CQD$.

**18.** In fig. 512, $\triangle ABC = \triangle XYZ$, prove that

quad. ABYX + quad. BCZY = quad. ACZX.

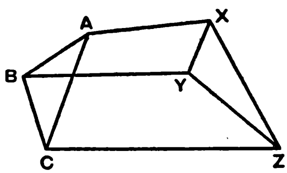



Fig. 512

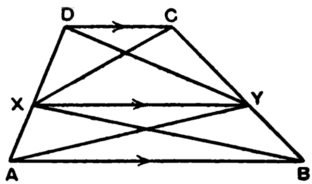



Fig. 513

**19.** In fig. 513, prove that $\triangle XBC = \triangle YAD$.

[20] P and Q are any points on the sides AB, DC of the trapezium ABCD in fig. 513. Prove that quad. XPYQ = $\triangle BXC$.

[21] ABCD is a parallelogram; P is any point on BC; DQ is the perpendicular from D to AP.

Prove that the area of ABCD = DQ . AP.

[22] ABCD is a parallelogram; a line through A cuts CB produced at P and cuts CD produced at Q. If $\triangle BAP = \triangle DAQ$, prove that AP = AQ. [Join AC.]

[23] BE, CF are medians of $\triangle ABC$ and cut at G. Prove that

quad. AEGF = $\triangle BGC$.

**24.** In fig. 514, D is the mid-point of BC. Prove that PQ bisects the area of $\triangle ABC$. [Join AD.]

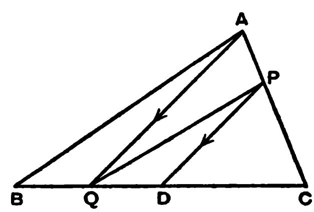



Fig. 514

**25.** P, Q are points on the sides CB, CD of the parallelogram ABCD such that PQ is parallel to BD. Prove that $\triangle ABP = \triangle ADQ$. [Join PD, BQ.]

[26] In fig. 515, prove that $\triangle APB = \triangle AQC$. [Join CP, BQ.]

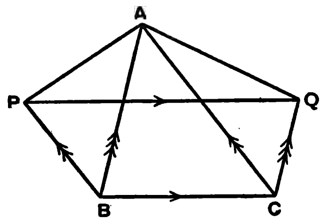



Fig. 515

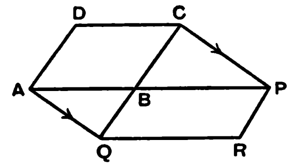



Fig. 516

27. In fig. 516, ABP, CBQ are straight lines and ABCD, PBQR are parallelograms. Prove that $ABCD = PBQR$. [Join AC, PQ.]

28. ABCD is a parallelogram; any line through A cuts DC at Y and cuts BC produced at Z. Prove that $\triangle BCY = \triangle DYZ$. [Prove that $\triangle DYZ = \triangle AYC$.]

*29. AB, DC are the parallel sides of a trapezium ABCD; AC cuts BD at K. If the line through K parallel to BA cuts AD at P, prove that $\triangle PBC = 2\triangle KAD$.

*30. E is a point on the side CD of the parallelogram ABCD, and CD is produced to F so that $DF = CE$; BE produced cuts AD produced at G and cuts the line through F parallel to AD at H. Prove that $AFHG = ABCD$.

Nos. 31–33 refer to fig. 517 in which ABCD is a parallelogram, and PQ, RS are parallel to AD, AB.

*31. Prove that
$\triangle SPA + \triangle SPQ = \triangle SPD$.

*32. Prove that
$\triangle APR + \triangle ASQ = \triangle ABD$.

*33. Prove that
$APKR - KSCQ = 2\triangle BKD$.

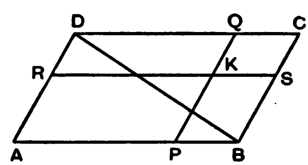



Fig. 517

*34. ABCD is a parallelogram; DC is produced to P; AP cuts BD at Q. Prove that $\triangle DQP - \triangle AQB = \triangle BCP$.

*35. The diagonals AC, BD of the quadrilateral ABCD cut at K. CA, DB are produced to P, Q respectively so that $AP = CK$ and $BQ = DK$. Prove that quad. $ABCD = \triangle KPQ$.

*36. P is a variable point inside a given equilateral triangle ABC; PX, PY, PZ are the perpendiculars from P to BC, CA, AB. Prove that $PX + PY + PZ$ is constant.

## CONSTRUCTION 9

**Construct a triangle equal in area to a given quadrilateral.**

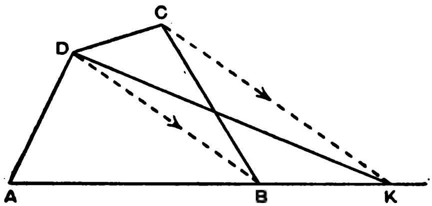



FIG. 518

**Given** a quadrilateral ABCD.

**To construct** a triangle equal in area to ABCD.

**Construction.** Join BD.

Through C draw CK parallel to DB to meet AB produced at K.

Join DK.

Then △AKD = quad. ABCD.

**Proof.** The △s DBK, DBC are on the same base DB and between the same parallels DB, CK,

∴ △DBK = △DBC.

Add to each △DBA,

∴ △AKD = quad. ABCD.

In order to construct a triangle equal in area to a given pentagon ABCDE, we proceed exactly as in Construction 9. This gives a quadrilateral AKDE, see fig. 519, equal in area to ABCDE. Repeating the process, we can then construct a triangle equal in area to AKDE.

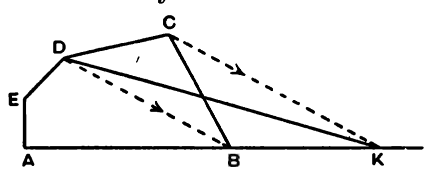



FIG. 519

The same method can be used for a polygon with any number of sides. Each repetition of the process gives an equivalent polygon in which the number of sides has been reduced by one.

## EXERCISE 52

[*Perform the constructions in this exercise without making calculations, whenever possible.*]

1. Draw a triangle ABC in which BC = 6 cm., CA = 5 cm., AB = 7 cm. Construct an acute-angled triangle KBC equal in area to $\triangle$ ABC and such that KB = 8 cm. Measure $\angle$ KBC.

[2] Copy $\triangle$ ABC in No. 1. Construct an equivalent triangle PBC such that PB = PC. Measure PB.

3. Copy $\triangle$ ABC in No. 1. Construct an equivalent triangle AQR in which AQ = 5·5 cm. and $\angle$ AQR = 70°. Measure QR. [First draw equivalent $\triangle$ ABQ, with AQ = 5·5 cm.]

[4] Draw a parallelogram ABCD in which AB = 6·5 cm., AD = 5 cm., $\angle$ BAD = 62°. Construct an equivalent parallelogram with sides equal to 6·5 cm., 6 cm., and measure one of its acute angles.

5. Draw a parallelogram ABCD in which AB = 4 cm., AD = 6 cm., $\angle$ BAD = 70°. Construct an equivalent parallelogram with sides equal to 5 cm., 5·8 cm., and measure an acute angle. [First draw equivalent ||gram ADKH with AH = 5 cm.]

6. Draw an equilateral triangle, side 3 in. Construct an equivalent rhombus in which each side is 2·5 in. long. Measure the acute angle of the rhombus.

[7] Draw a regular hexagon ABCDEF, such that AB = 4 cm. Construct an equivalent rectangle AEPQ. Measure EP and find the area of the hexagon.

8. Draw a triangle with sides 5 cm., 6 cm., 8 cm. Construct an equivalent acute-angled triangle with two of its sides 5·6 cm., 6·5 cm. Measure the third side.

[9] Construct a parallelogram ABCD of area 8 sq. in. such that AC = 5 in., BD = 4 in. Measure AB and BC.

10. Construct a triangle ABC of area 4·5 sq. cm., given that the radius of the circle ABC is 5 cm. and that BC = 3 cm. Explain shortly your method.

11. Draw quad. ABCD given AB = 3 cm., BC = 5 cm., CD = 6 cm., DA = 4 cm., BD = 5 cm. Construct a point K on AB produced such that $\triangle$ DAK = quad. ABCD. Find the area of ABCD.

[12] Draw a quadrilateral ABCD in which AB = 6 cm., BC = 5 cm., CD = 4 cm., $\angle$ ABC = 110°, $\angle$ BCD = 95°. Construct an equivalent triangle ABP such that P lies on BC produced and find the area of ABCD.

13. Draw any triangle ABC and take a point D on BC produced. Construct a point Q on AB such that $\triangle$ BQD is equal to $\triangle$ BAC. State your method and prove that it is correct.

[14] Draw any parallelogram ABCD and take a point E on AB produced. Construct a point P on AD such that the parallelogram PAEQ is equivalent to ABCD. State your method and prove that it is correct. [$\triangle$ EAP = $\triangle$ ABD.]

*15. Draw any triangle ABC and take a point D inside it; join DB, DC. Construct a point P on AC such that $\triangle$ APB equals the area of the re-entrant quadrilateral ABDC. State your method and prove that it is correct.

16. Draw a quadrilateral ABCD in which $\triangle$ ABC is greater than $\triangle$ ADC. Construct a line through A which bisects the area of ABCD. [Find K on BC produced such that

$\triangle$ ABK = quad. ABCD.]

[17] Draw a parallelogram ABCD. Construct points P, Q on BC, CD such that AP, AQ divide ABCD into three parts of equal area. State your method and prove that it is correct.

18. (i) In fig. 520, Q is any point on AC. Use the fact that the area of a parallelogram is bisected by a diagonal to prove that XQRD = PQYB.

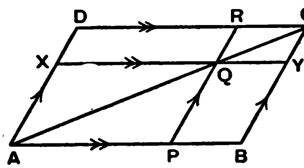


FIG. 520

(ii) Draw a parallelogram XQRD in which XQ = 6 cm., XD = 4 cm., $\angle$DXQ = 70°. Construct an equivalent parallelogram in which one side is 5 cm. and one angle is 70°. Measure the other side.

[19] Draw an equilateral triangle, side 3 in. Construct an equivalent rectangle having one side 2·5 in. long. Measure the other side. [Use the fact stated in No. 18 (i).]

20. Draw a parallelogram with sides 2·1 in., 2·9 in., and one angle 72°. Construct a rectangle equal in area to the parallelogram and with one side 2 in. long. Measure the other side.

[21] Construct a rhombus having each of its sides equal to 5 cm. and equal in area to a rectangle whose sides are 6 cm., 3 cm. long. Measure the acute angle of the rhombus.

*22. Draw a triangle ABC in which BC = 6 cm., CA = 5 cm., AB = 7 cm. Take a point P on AC such that AP = 2 cm. Construct a point Q on BC such that PQ bisects the area of $\triangle$ ABC. Measure CQ. [See Ex. 51, No. 24.]

## PYTHAGORAS' THEOREM

Fig. 521 represents part of a floor tiled with equal tiles, each in the shape of an isosceles right-angled triangle.

Can you see any connection between the area of the square on the hypotenuse and the sum of the areas of the squares on the other two sides of one of these right-angled triangles?

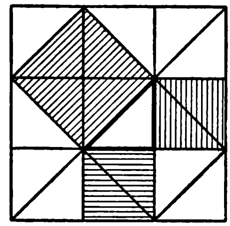
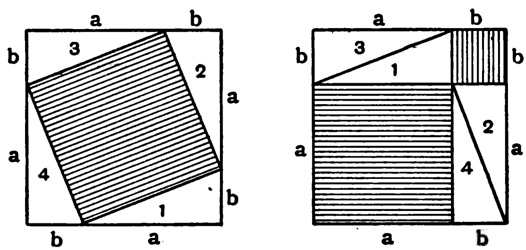


FIG. 521

If the right-angled triangle is not isosceles, the connection can be examined as follows:—

Fold a rectangular sheet of paper and then fold it again so as to get four rectangular layers and then cut out four *duplicate* right-angled triangles.




(i) FIG. 522 (ii)

Suppose the hypotenuse is $c$ in., and that the other two sides are $a$ in., $b$ in. Draw (or cut out) two equal squares of side $(a+b)$ in. and arrange the triangles on the top of each square, firstly as in fig. 522 (i), secondly as in fig. 522 (ii). In fig. 522 (i), the uncovered shaded area is the square on the hypotenuse of the triangle labelled 1; in fig. 522 (ii), the uncovered shaded area is made up of the squares on the other two sides of the triangle labelled 1.

What does this experiment suggest?

If fig. 522 is used to obtain a *proof* of the fact suggested, it is necessary to show that the shaded figures are square; this is not difficult.

## Perigal's Dissection

Fig. 523 shows a method of cutting up a square by means of which the squares on the two sides containing the right angle of a right-angled triangle can be made to cover exactly the square on the hypotenuse: a kind of jig-saw puzzle.

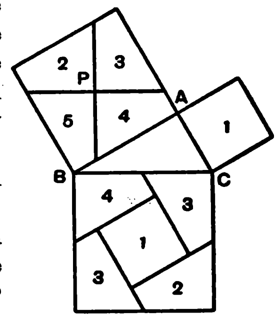



FIG. 523

Draw a triangle ABC, right-angled at A, on thin cardboard or stiff paper, and construct the three squares on its sides. Take the centre P of the square on AB and draw through P lines parallel and perpendicular to BC. Cut out the squares on AB, AC and cut up the square on AB into the four parts indicated. Arrange the pieces to cover the square on BC.

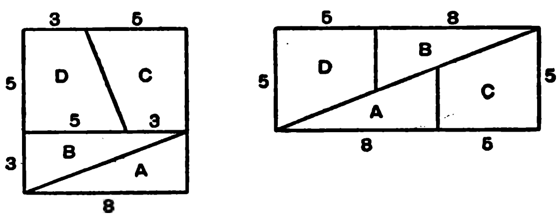



FIG. 524

It must be realised that a *proof* is essential. If one relies merely on appearances, it is easy to make mistakes. For example, draw on squared paper with $\frac{1}{5}$ inch as unit, a square, side 8 units long, and a rectangle 13 units long, 5 units high. Cut up the square into the four pieces shown and fit them on to the rectangle in the positions indicated. Do they cover up the rectangle? If so, this suggests that the area of the square is equal to that of the rectangle. Is this true?

Pythagoras lived in the sixth century B.C. Special cases of his theorem were known to the Egyptians much earlier.

at least by 1000 B.C., as their surveyors made use of the fact that a triangle with sides 3, 4, 5 units is right-angled. Probably Pythagoras learnt this from Thales, who urged him to visit Egypt where he spent many years. Pythagoras' method of proof is not known; the proof on p. 282 is due to Euclid, who lived 300 years later and wrote the greatest text-book of all time, known as Euclid's *Elements*.

### Examples for Oral Discussion

1. In fig. 525, ABHK, BCPQ are squares. Prove that

(i) $\angle HBC = \angle ABQ$; (ii) $\triangle HBC \equiv \triangle ABQ$.

If the triangle HBC is rotated clockwise about B through a right angle, what is its new position? What can you say about the directions of CH and AQ?

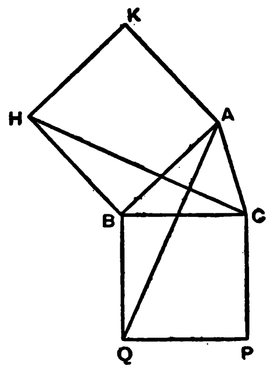


FIG. 525

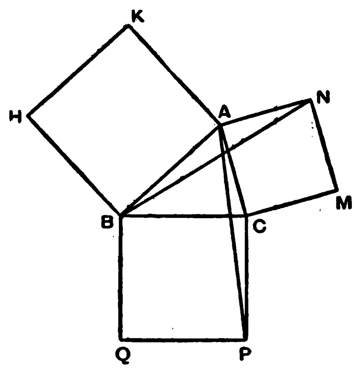


FIG. 526

2. In fig. 526, ABHK, ACMN, BCPQ are squares. Make a sketch of the figure. By joining two points in your sketch obtain a triangle congruent to $\triangle ACP$. Similarly, obtain a triangle congruent to $\triangle ABN$. Give reasons.

In figs. 527, 528, ABHK, BCPQ are squares. In fig. 527, $\angle BAC$ is acute; in fig. 528, $\angle BAC$ is a right angle. *Make sketches of both figures* and use them for Nos. 3–5.

3. Name 3 points which are collinear in fig. 528 but not collinear in fig. 527. Give reasons.

4. (i) By joining points in your sketches of figs. 527, 528, obtain three triangles with base BQ which are equal to half the rectangle BQYX.

(ii) Similarly, obtain triangles with base HB which are equal to half the square HBAK. Are they the same for both figures?

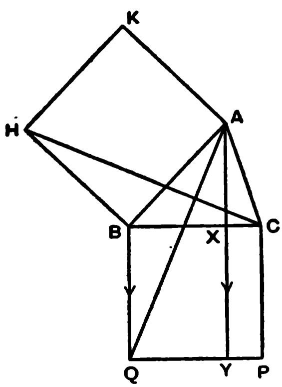



Fig. 527

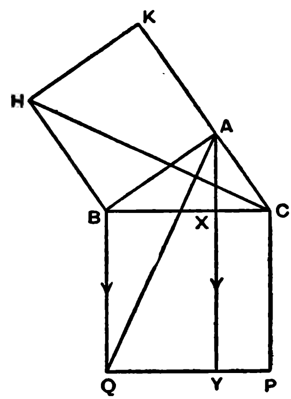



Fig. 528

5. (i) In your sketch of fig. 527, construct a rectangle HBDE whose area is double that of $\triangle$HBC.

(ii) Use the fact that $\triangle$ABQ $\equiv$ $\triangle$HBC to find a rectangle in your sketch of each figure which is equal in area to BQYX. Is it the same rectangle in the two figures?

(iii) Construct in the same way rectangles which have the same area as CPYX in fig. 527 and in fig. 528.

(iv) What can you now say about the area of the square BCPQ in fig. 527 and in fig. 528?

The results obtained in these examples may be summarised as follows:—

If in $\triangle$ABC, $\angle$BAC is **acute**, see fig. 529, the area of the square on BC is less than the sum of the areas of the squares on AB and AC by the sum of the areas of the rectangles ADEK, AFGN.

Explain why rect. ADEK = rect. AFGN.

Prove that the area of each of the rectangles ADEK, AFGN is AB . AC cos BAC.

Hence with the usual notation for $\triangle$ABC, if $\angle$BAC is acute,

$$a^2 = b^2 + c^2 - 2bc \cos A.$$

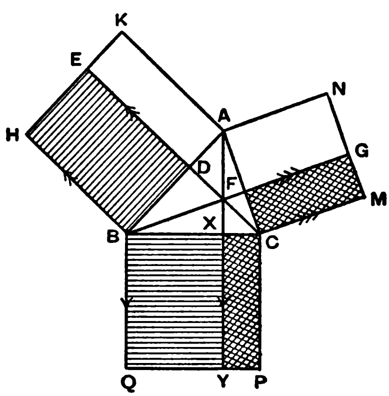



Fig. 529

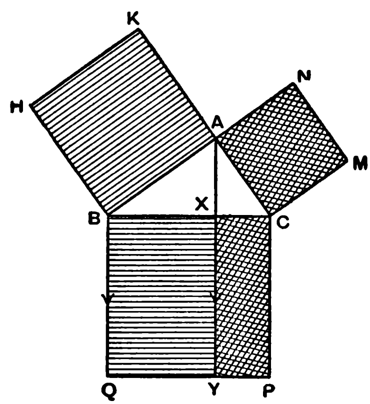



Fig. 530

If in $\triangle$ABC, $\angle$BAC is a **right angle**, see fig. 530, the square on BC can be divided into two rectangles whose areas are equal to those of the squares on AB and AC respectively. This provides a proof of **Pythagoras' theorem.**

**In a right-angled triangle, the area of the square on the hypotenuse is equal to the sum of the areas of the squares on the other two sides.**

*Similarly, if you draw a figure like fig. 529, but so that $\angle$ BAC is obtuse, you will see that the square on BC can be divided into two rectangles, the area of one of which is greater than the area of the square on AB, and the area of the other is greater than the area of the square on AC.

Therefore in $\triangle$ABC, if $\angle$BAC is **obtuse**, the area of the square on BC is **greater** than the sum of the areas of the squares on AB and AC by the sum of the areas of two rectangles, the area of each of which is AB . AC cos (180° − $\angle$BAC).

Hence with the usual notation for $\triangle$ABC, if $\angle$BAC is obtuse,

$$a^2 = b^2 + c^2 + 2bc \cos (180° - \mathrm{A}).$$

On p. 79, a definition was given for the cosine of an *acute* angle only. It is *convenient* to define the cosine of an obtuse angle to be *minus* the cosine of its supplement, *e.g.* by definition cos 150° = − cos 30°, cos 110° = − cos 70°, etc. Therefore by definition, if $\angle$BAC is obtuse,

$$\cos \mathrm{A} = -\cos (180° - \mathrm{A}).$$

Hence in $\triangle$ABC, if $\angle$BAC is obtuse,

$$\mathbf{a^2 = b^2 + c^2 - 2bc \cos A}.$$

Thus the same formula holds for obtuse-angled triangles as for acute-angled triangles.

**Acute-angled, Right-angled, and Obtuse-angled Triangles.** If the lengths of the sides of a triangle are given, it is easy to find out whether the triangle is acute-angled or not:

If $\angle$BAC is acute, then $BC^2 < BA^2 + AC^2$.
If $\angle$BAC is a right angle, then $BC^2 = BA^2 + AC^2$.
If $\angle$BAC is obtuse, then $BC^2 > BA^2 + AC^2$.

These tests cover every possibility, and therefore the converse statements are true.

**Oral Example.** Is a triangle obtuse-angled if the lengths of its sides are (i) 8 cm., 9 cm., 12 cm.?
(ii) 7 cm., 12 cm., 14 cm.?

* This paragraph may be omitted at a first reading.

**Areas associated with Pythagoras' Theorem.** This theorem is fundamentally a property of areas, although its usefulness lies mainly in its application to the calculations of lengths and to Trigonometry. In order to make sure that the theorem is understood, it is desirable that in the first instance some applications should be discussed in which the reference is to *actual areas*; there is no risk of the arithmetical aspect of the theorem being overlooked. The following examples are intended to serve this purpose and to emphasise the importance of the theorem, which, with the notation of fig. 530, p. 270, can be expressed in the form $BA^2 = BX.BC$.

**Examples for Oral Discussion.** *Make a sketch of fig.* 530, *without the shading, for each of Nos.* 1–3.

1. If $\angle BAC = 90°$, $AB = 4$ in., $BC = 5$ in., show on your sketch the areas in succession of sq. AH, rect. BY, rect. CY, sq. AM. Find the lengths of BX, CX, AC.

2. If $\angle BAC = 90°$, $AC = 6$ in., $BC = 10$ in., show on your sketch the areas in succession of sq. AM, rect. CY, rect. BY, sq. AH. Find the lengths of CX, XB, AB.

3. If $\angle BAC = 90°$, $AB = 12$ in., $AC = 5$ in., show on your sketch the areas in succession of sq. AH, sq. AM, rect. BY, rect. CY. What is the length of BC?

4. In fig. 530, if $\angle BAC = 90°$, $AC = 2$ in., $BC = 3$ in., find the area of the square ABHK. Calculate the length of AB, correct to $\frac{1}{100}$ in.

5. *Construct* a square of area 10 sq. in. Then find by measurement the value of $\sqrt{10}$ approximately.

6. *Construct* a square of area 7 sq. in. Then find by measurement the value of $\sqrt{7}$ approximately.

7. In fig. 530, if $\angle BAC = 90°$, $\angle AXC = 90°$, $AX = 4$ in., $XC = 3$ in., find the area of rect. XP, and the lengths of CP, BX, BA.

8. In fig. 530, if $\angle BAC = 90°$, $\angle ACB = 60°$, prove that sq. ABHK is 3 times sq. ACMN. [Note that $\triangle BAC$ is half an equilateral triangle, see p. 121.]

What are the values of sin 60° and tan 60°?

**Numerical Applications**

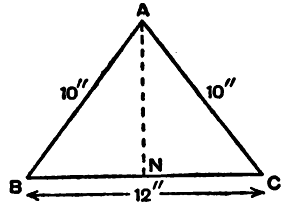


FIG. 531

**Example.** ABC is a triangle such that

AB = AC = 10 in., BC = 12 in.

Calculate the area of △ABC.

If N is the mid-point of BC, AN is perpendicular to BC, and BN = 6 in.

Let AN = $h$ in.

Since ∠ANB is a right angle,

$$h^2 + 6^2 = 10^2;$$
$$\therefore\ h^2 = 100 - 36 = 64;$$
$$\therefore\ h = 8.$$

∴ the altitude AN of △ABC is 8 in.

$$\therefore \text{ area of } \triangle ABC = \tfrac{1}{2} \times 12 \times 8 \text{ sq. in.}$$
$$= 48 \text{ sq. in.}$$

NOTE.—For square-root tables see p. xiv.

## NUMERICAL EXAMPLES

### EXERCISE 53

[*Give answers which are approximate, correct to* 3 *figures. Unless otherwise stated, the unit of length in each diagram is* 1 *in.*]

Nos. 1–9 refer to fig. 532.

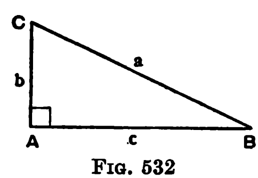


FIG. 532

1. $b = 8$, $c = 15$, find $a$.

[2] $b = 5$, $c = 12$, find $a$.

3. $b = 2$, $c = 3$, find $a$.

4. $a = 25$, $b = 15$, find $c$.

[5] $a = 6$, $c = 5$, find $b$.

6. ∠C = 45°, $a = 4$, find $b$. 7. ∠C = 60°, $a = 6$, find $b$ and $c$.

[8] ∠C = 60°, $b = 4$, find $a$ and $c$.

*9. AB − AC = 1 in., BC − AB = 2 in., find AB.

10. A 20-foot ladder is leaning against a wall. The end on the ground is 12 ft. from the wall. How high up the wall does the ladder reach?

[11] A ladder just reaches the top of a wall 18 ft. high when the end on the ground is 8 ft. from the wall. Find the length of the ladder.

12. A man travels 15 miles due north and then 5 miles due east. How far is he from his starting-point?

[13] The sides of a rectangle are 5 in., 7 in.; find the length of a diagonal.

14. Find the side of a rhombus whose diagonals are 6, 10 cm.

[15] An aeroplane heads north-west at 150 miles an hour and is carried north-east by a wind at 60 miles an hour. Find the distance from the starting-point after 1 hour.

16. In fig. 533, if AD = 10 cm., AB = 8 cm., BC = 4 cm., find the length of CD.

[17] In fig. 533, if AD = 8 in., BC = 3·5 in., CD = 7·5 in., find the length of AB.

18. PQRS is a quadrilateral in which ∠PQR and ∠PRS are right angles. If PQ = 12 cm., QR = 9 cm., RS = 8 cm., find the length of PS and the area of PQRS.

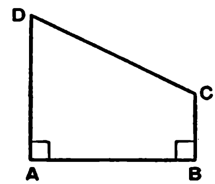



Fig. 533

[19] ABCD is a quadrilateral in which ∠B = ∠D = 90°. If AB = 6 in., BC = 8 in., CD = 5 in., find the length of AD and the area of ABCD.

20. AD is an altitude of the acute-angled triangle ABC. If AB = 20 in., BD = 16 in., DC = 5 in., find the length of AC.

[21] In △ABC, ∠B = 90°, AB = 8 cm., AC = 17 cm.; D is a point on BC such that BD = 5 cm. Find the area of △ADC.

22. In △ABC, ∠C = 90°, AC = 3 in., AB = 8 in. Find the length of the median AX.

23. In △ABC, AB = AC = 13 in., BC = 10 in. Find the area of △ABC and the length of the perpendicular from C to AB.

[24] Repeat No. 23 if AB = AC = 10 in., BC = 12 in.

[25] PQ is a chord of a circle, centre O, radius 6 cm.; PQ = 8 cm. Find the length of the perpendicular from O to PQ.

26. A gun, whose effective range is 9000 yd., is 5000 yd. from a straight railway. What length of the railway is commanded by the gun?

[27] Construct a square of area 13 sq. in. Measure the side of the square.

28. Construct a square of area 11 sq. in. Measure the side of the square.

29. In fig. 534, find the distance of D from A.

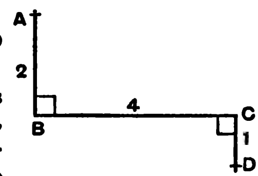


Fig. 534

30. Mark on squared paper the points whose co-ordinates are (1, 2) and (3, 5), and calculate the distance between them.

[31] Prove that the points whose co-ordinates are (5, 11), (6, 10), (7, 7) lie on a circle whose centre is (2, 7), and find its radius.

32. Prove that the triangle whose sides are 37 cm., 35 cm., 12 cm. is right-angled, and find its area.

33. ABCD is a quadrilateral in which AB = 9 in., BC = 12 in., AD = 25 in., CD = 20 in., $\angle$ABC = 90°. Find the area of ABCD.

34. Find out whether the triangle whose sides are 5 cm., 6 cm., 8 cm. is acute-angled or obtuse-angled.

[35] Repeat No. 34 for the triangle whose sides are 7 cm., 10 cm., 12 cm.

36. Prove that the points whose co-ordinates are (0, 0), (7, 17), (12, 5) are the vertices of an isosceles right-angled triangle.

37. In fig. 535, O is the centre of the circular arc ADB. Find the radius $r$ in. of the circle. [Note that OC = $(r-8)$ in.]

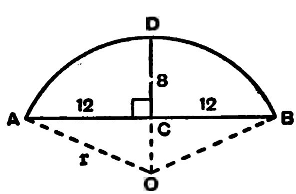


Fig. 535

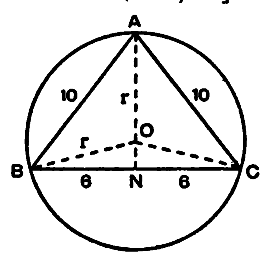


Fig. 536

38. In fig. 536, AB = AC = 10 in., BC = 12 in. Find the radius $r$ in. of the circle ABC.

[39] In $\triangle$ ABC, AB = AC = 10 in., BC = 16 in. Find the radius of the circle which passes through A, B, C.

40. In $\triangle$ ABC, AB = 4 in., BC = 5 in., $\angle$B = 45°. Find AC.

41. In $\triangle$ ABC, AB = 8 in., BC = 3 in., $\angle$B = 60°. Find AC.

*42. AD is an altitude of $\triangle$ ABC. If AB = 5 cm., BC = 9 cm., CA = 7 cm., prove that $\angle$BAC is obtuse and find the lengths of BD and AD. [Let BD = $x$ cm., AD = $h$ cm., and use Pythagoras twice.]

*43. AD, BC are two vertical poles, D and C being the ends on the ground which is level; AC = 12 ft., AB = 10 ft., BC = 3 ft.; find the length of AD.

*44. AD is an altitude of $\triangle$ ABC in which $\angle$B and $\angle$C are acute. If BD = $x^2$ cm., DC = $y^2$ cm., AD = $xy$ cm., find expressions for AB$^2$ and AC$^2$ in terms of $x$ and $y$, and prove that $\angle$BAC is a right angle.

*45. Prove that the triangle whose sides are of lengths $2xy$, $x^2 - y^2$, $x^2 + y^2$ in. is right-angled. Also, if $a^2 + ab + b^2 = c^2$, prove that the three triangles obtained by putting (i) $x = c$, $y = a$, (ii) $x = c$, $y = b$, (iii) $x = a + b$, $y = c$, are all equal in area.

Prove also that if $a = 2mn + n^2$, $b = m^2 - n^2$, $c = m^2 + mn + n^2$, then $a^2 + ab + b^2 = c^2$. What result is obtained by putting $m = 2$, $n = 1$?

*46. Fig. 537 represents a horizontal section of the axles and wheels of a carriage with front wheels at full lock. The inner back wheel is travelling round a circle of radius 9 ft. Find the radius of the track traced out by the outer front wheel.

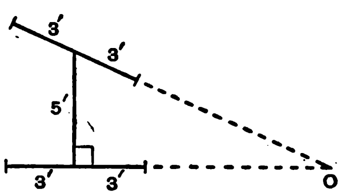



FIG. 537

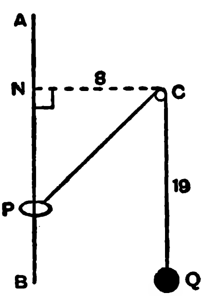



FIG. 538

*47. AB is a fixed vertical wire and C is a small fixed peg, 8 in. from AB, see fig. 538. PCQ is a string 3 ft. long supporting a marble Q and attached to a small ring P which can slide on AB. Initially Q is 19 in. below C. If Q is pulled down 7 in., how much does P rise?

*48. A small heavy object L is raised to a platform AE by a chain passing over a small pulley B and continuing along the jib BA down to a drum A on to which it is wound, see fig. 539. L is raised by first winding up the chain on to A and then rotating in a vertical plane about A. Find the least length of chain that must be wound on to the drum so that L may clear the corner E of the platform.

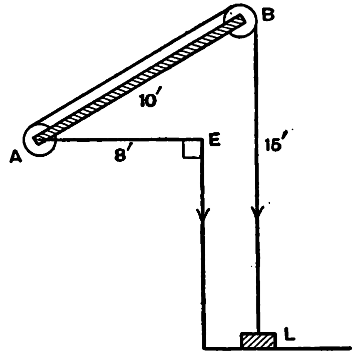



FIG. 539

## SOLID GEOMETRY

### Examples for Oral Discussion

CUBOID

**Standard Notation for Cuboid.** The notation ABCDEFGH for a cuboid means that ABCD, EFGH are parallel faces joined by *edges* AE, BF, CG, DH.

1. A room is 17 ft. long, 14 ft. wide, 9 ft. high. Find the distance from a corner A of the floor to the opposite corner G of the ceiling. Let the diagonal AC of the floor be $x$ ft. long, and the diagonal AG of the room be $y$ ft. long.

(i) Find $x^2$ from $\triangle$ABC.

(ii) Find $y^2$ from $\triangle$ACG.

*Notice that it is unnecessary to find the value of $x$.*

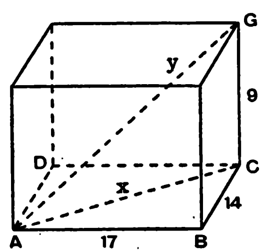



FIG. 540

2. Find the length of a diagonal of a cuboid whose edges are 7 cm., 5 cm., 3 cm.

## Right Pyramid

3. If, in fig. 541, ABCD is a *rectangle* whose diagonals intersect at N, and if V is any point on the perpendicular through N to the plane ABCD, prove that

$$VA = VB = VC = VD.$$

Prove that $\triangle VNA \equiv \triangle VNB$.

Fig. 541 represents a **right** pyramid, vertex V, whose base is the **rectangle** ABCD; the length of VN is called the *height* of the pyramid.

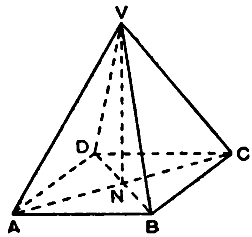



Fig. 541

4. In fig. 541, which represents a right pyramid on a rectangular base ABCD, if AB = 10 in., BC = 6 in., VA = 12 in., find the height VN.

Make a sketch of fig. 541 and mark on it the mid-point P of AB.

(i) Find $VP^2$ from the right-angled triangle VPA.
(ii) Find VN from the right-angled triangle VNP.

5. If, in fig. 542, ABC is an *equilateral triangle* whose medians intersect at N, and if V is any point on the perpendicular through N to the plane ABC, prove that VA = VB = VC.

(i) Explain why NA = NB.
(ii) Explain why

$$\triangle VNA \equiv \triangle VNB.$$

Fig. 542 represents a **right** pyramid, vertex V, whose base is the **equilateral triangle** ABC: the length of VN is called the *height* of the pyramid.

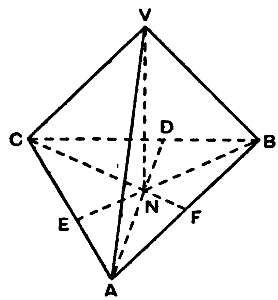



Fig. 542

Since N is the median-centre or centroid of $\triangle ABC$, $AN = \frac{2}{3}AD$, see p. 192. By using this fact, the length of AN can be found when the length of AB is given.

**6.** In fig. 542, which represents a right pyramid whose base is an equilateral triangle ABC, if $BC = 6$ cm., $VA = 8$ cm., find the height VN.

(i) Find $AD^2$ from the right-angled triangle ADB.
(ii) Complete: $AN = \frac{2}{3}AD$, $\therefore$ $AN^2 = \ldots = \ldots$
(iii) Find VN from the right-angled triangle VNA.

*Alternatively*, in place of (i) and (ii), find $AN^2$ by using the fact that $\triangle ANF$ is "half an equilateral triangle," see p. 121.

## SPHERE

**7.** Fig. 543 represents a sphere, centre O, and APB is a section of the surface made by any plane. NOS is the diameter of the sphere perpendicular to the plane APB and cuts this plane at K. Prove that the section APB is a circle, centre K.

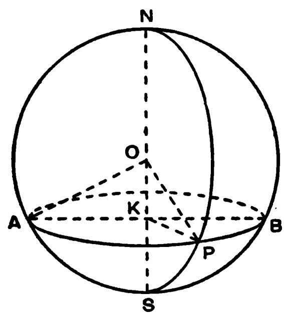



FIG. 543

(i) If P is any point on the plane section APB, explain why $\triangle OKA \equiv \triangle OKP$.
(ii) Complete the proof.

**8.** In fig. 543, if the diameter of the sphere is 20 cm., and if the distance of the plane APB from the centre of the sphere is 6 cm., find the radius of the circular section APB.
Use the right-angled triangle OKA.

# NUMERICAL EXAMPLES

## EXERCISE 54

[*Give answers which are approximate, correct to three figures.*]

1. A room is 20 ft. long, 16 ft. wide, 8 ft. high. Find the length of the line joining a corner of the floor to the opposite corner of the ceiling.

[2] Find the diagonal of a cube whose edge is 5 in.

3. What is the length of the longest straight rod, measuring a whole number of inches, that can be put into a rectangular box whose internal measurements are 6 ft. by 5 ft. by 4 ft.?

[4] The edge of a cube is 6 in. Find the distance between the centres of two adjacent faces.

5. AB runs due east, BC runs due north, CD is vertical. If AB = 12 ft., BC = 6 ft., CD = 12 ft., find the distance of D from A.

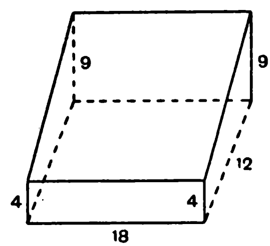


FIG. 544

6. Fig. 544 represents a closed box with a sloping lid; the base is a horizontal rectangle, 18 in. by 12 in., and the side faces are vertical. Find the area of the lid.

[7] ABCD is the rectangular floor of a room of height 10 ft.; E is the corner of the ceiling above A. If AB = 24 ft., BC = 20 ft., find the area of △ EBC.

8. Fig. 545 represents a circular cone, vertex V; VN is the *axis* of the cone, *i.e.* the line joining the vertex to the centre of the base. If the diameter of the base is 10 cm., and if the height VN of the cone is 12 cm., find the slant height VA of the cone.

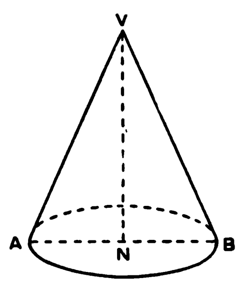


FIG. 545

9. If the slant height of a circular cone is 10 in. and the diameter of its base is 8 in., find the height of the cone.

[10] If the slant height of a circular cone is 6 in. and if the height is 4 in., find the diameter of the base.

11. The base of a right pyramid of height 4 in. is a square whose diagonal is 6 in. Find the length of a sloping edge.

12. The base ABCD of a right pyramid, vertex V, is a square of side 6 cm. and the length of the perpendicular from V to AB is 5 cm. Find the height of the pyramid and the length of a sloping edge.

[13] The base of a right pyramid of height 8 in. is a square whose side is 12 in. Find the area of a side face and the length of a sloping edge.

14. If, in fig. 541, p. 278, which represents a right pyramid on a rectangular base, AB = 8 cm., BC = 6 cm., VA = 13 cm., find the height VN of the pyramid.

[15] If, in fig. 541, AB = 6 in., BC = 4 in., VN = 5 in., find the length of VA.

*16. If, in fig. 542, p. 278, which represents a right pyramid whose base is an equilateral triangle ABC, AB = 9 cm., VA = 6 cm., find the height VN of the pyramid.

*17. If, in fig. 542, AB = 12 in., VN = 4 in., find the length of VA.

18. A hemispherical bowl, internal diameter 20 in., contains some water. If the greatest depth of the water is 4 in., find the radius of the circle formed by the water-surface.

19. A circle of *radius* 2 in. is drawn on the surface of a sphere of *diameter* 6 in. Find the distance of the plane of the circle from the centre of the sphere.

*20. ABCD is the base of a cuboid, and AP, BQ, CR, DS are the edges at right angles to the base; AB = 8 in., BC = 6 in., AP = 5 in.; H is the mid-point of BC, K is the mid-point of PQ. Find the distances (i) HP, (ii) HK, (iii) DK.

*21. ABCD is a rectangle; AB = 6 in., BC = 8 in.; it is folded about BD so that the planes of the two parts are at right angles. Find the new distance of A from C.

*22. Three points A, B, C on the surface of a sphere of diameter 8 in. are the vertices of an equilateral triangle of side 6 in. Find the distance of the plane of the triangle from the centre of the sphere.

## THEOREM 40 (Pythagoras' Theorem)

**In any right-angled triangle, the area of the square on the hypotenuse is equal to the sum of the areas of the squares on the sides containing the right angle.**

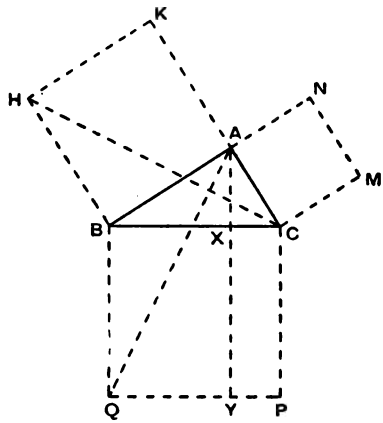



FIG. 546

**Given** a triangle ABC in which $\angle$ BAC = 1 right angle.

**To prove that** sq. on BC = sq. on BA + sq. on AC.

**Construction.** Draw the squares BCPQ, ABHK, ACMN, outside the triangle ABC.
Through A draw a line AXY parallel to BQ, and therefore also parallel to CP, to meet BC at X and QP at Y.
Join AQ, CH.

**Proof.** $\angle$ BAC = 1 rt. $\angle$ *given,*
$\angle$ BAK = 1 rt. $\angle$ *$\angle$ of square.*

$\therefore$ $\angle$ BAC + $\angle$ BAK = 2 rt. $\angle$s.

$\therefore$ KA and AC are in the same straight line.

$\angle$ HBA = $\angle$ CBQ *rt. $\angle$s, $\angle$s of squares*

Add to each $\angle$ ABC,

$\therefore$ $\angle$ HBC = $\angle$ ABQ.

In the $\triangle$s HBC, ABQ,

$$\begin{aligned} HB &= AB && \textit{sides of square,} \\ CB &= QB && \textit{sides of square,} \\ \angle HBC &= \angle ABQ && \textit{proved.} \end{aligned}$$

$\therefore$ $\triangle$s $\begin{matrix} HBC \\ ABQ \end{matrix}$ are congruent SAS.

$\therefore$ area HBC = area ABQ.

But $\triangle$HBC and sq. HBAK are on the same base HB and between the same parallels HB, KAC,

$\therefore$ area HBC = $\frac{1}{2}$ area HBAK.

Also $\triangle$BQA and rect. BQYX are on the same base BQ and between the same parallels BQ, AXY,

$\therefore$ area BQA = $\frac{1}{2}$ area BQYX.

$\therefore$ area HBAK = area BQYX,

that is, sq. on BA = rect. BQYX.

Similarly, by joining AP, BM it can be proved that

sq. on AC = rect. CPYX.

$\therefore$ sq. on BA + sq. on AC = rect. BQYZ + rect. CPYX

= sq. on BC.

**Corollary. If the triangle ABC is right-angled at A and if AX is an altitude, then**

$$BA^2 = BX \, . \, BC \quad \text{and} \quad CA^2 = CX \, . \, CB.$$

Area of square on AB = area of rect. BQYX

$= BX \, . \, BQ.$

But $BQ = BC$ *sides of square,*

$\therefore$ $BA^2 = BX \, . \, BC.$

Similarly, area of square on AC = area of rect. CPYX.

$\therefore$ $CA^2 = CX \, . \, CP = CX \, . \, CB.$

## Trigonometrical Proof of Pythagoras' Theorem

Let ABC be a triangle right-angled at A, and AX an altitude.

With the notation of fig. 547;

from $\triangle$BAC, $\cos B = \frac{c}{a}$;

from $\triangle$AXB, $\cos B = \frac{p}{c}$;

$\therefore \frac{c}{a} = \frac{p}{c}$. $\quad \therefore c^2 = ap$.

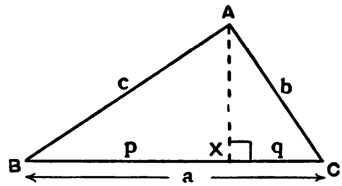


Fig. 547

Similarly, $b^2 = aq$. $\therefore b^2 + c^2 = ap + aq = a(p+q) = a^2$.

Note. The previous proof, pp. 282–3, depended on showing that $BA^2 = BX \cdot BC$, that is $c^2 = pa$: this was done by proving that the areas of the square and rectangle represented by these expressions are equal. The trigonometrical proof depends on the fact that equiangular triangles are the same shape.

## Trigonometrical Equivalent of Pythagoras' Theorem

With the notation of fig. 547, $b = a \sin B$ and $c = a \cos B$.

$$\therefore a^2(\sin B)^2 + a^2(\cos B)^2 = a^2,$$

$$\therefore (\sin B)^2 + (\cos B)^2 = 1.$$

**This is written in the form $\sin^2 B + \cos^2 B = 1$.**

**By means of this formula, the cosine of an acute angle can be calculated when the sine is given, and *vice versa*.**

For example, $\cos 60° = \frac{1}{2}$, see p. 121,

$$\therefore \sin^2 60° = 1 - \cos^2 60° = 1 - \frac{1}{4} = \frac{3}{4}.$$

But sin 60° is positive, $\therefore \sin 60° = \frac{\sqrt{3}}{2}$.

**The Converse of Pythagoras' Theorem.** The difficulty in remembering the construction used in Theorem 41 is diminished if the reason for it is understood. We cannot apply Pythagoras' theorem to the given $\triangle$ABC which we have to prove is *right-angled.* We therefore construct a *right-angled* $\triangle$XYZ as much like $\triangle$ABC as possible, and start the proof by applying Pythagoras' theorem to $\triangle$XYZ.

## THEOREM 41

**If the area of the square on one side of a triangle is equal to the sum of the areas of the squares on the other sides, then the angle contained by these sides is a right angle.**

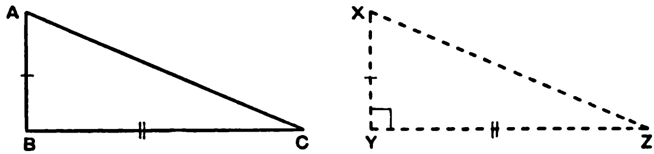



FIG. 548

**Given** a triangle ABC in which

sq. on AC = sq. on AB + sq. on BC.

**To prove that** $\angle B = 1$ right angle.

**Construction.** Draw a triangle XYZ in which

$$XY = AB.$$
$$\angle Y = 1 \text{ rt. } \angle.$$
$$YZ = BC.$$

**Proof.** Since $\angle Y = 1$ rt. $\angle$, $XZ^2 = XY^2 + YZ^2$ *Pythagoras.*

But $XY = AB$, $YZ = BC$ *constr.*

$\therefore$ $XZ^2 = AB^2 + BC^2$,

but $AC^2 = AB^2 + BC^2$ *given,*

$\therefore$ $XZ^2 = AC^2$,

$\therefore$ $XZ = AC$.

$\therefore$ in $\triangle$s ABC, XYZ

$AB = XY$ *constr.,*
$BC = YZ$ *constr.,*
$AC = XZ$ *proved.*

$\therefore$ $\triangle$s $\begin{matrix} ABC \\ XYZ \end{matrix}$ are congruent SSS.

$\therefore$ $\angle B = \angle Y$.

But $\angle Y = 1$ rt. $\angle$ *constr.*

$\therefore$ $\angle B = 1$ rt. $\angle$.

## EXERCISE 55

1. ABCD is a square. Prove that $AC^2 = 2AB^2$.

2. AD is an altitude of the equilateral triangle ABC. Prove that $4AD^2 = 3BC^2$. [Let $BC = 2x$ units.]

[3] ABCD is a quadrilateral in which $AB = BC = 2CD$ and $\angle ABD = \angle BCD = 1$ rt. $\angle$. Prove that $AD = 3CD$.

4. APQB is a straight line such that $AP = \frac{1}{4}AB$ and $BQ = \frac{1}{3}BA$. Prove that the triangle whose sides are equal to AP, PQ, QB respectively is right-angled.

Given a piece of thin string, show how to form a right angle.

5. AD is an altitude of $\triangle ABC$. If $\angle C = 45°$, prove that $AB^2 = BD^2 + DC^2$.

[6] AD is an altitude of $\triangle ABC$. Prove that

$$AB^2 + CD^2 = AC^2 + BD^2.$$

7. ABCD is a quadrilateral in which $\angle B = \angle D = 1$ rt. $\angle$. Prove that $AB^2 - AD^2 = CD^2 - CB^2$.

8. In fig. 549, AKD is perpendicular to BDC. Prove that $KB^2 - KC^2 = AB^2 - AC^2$.

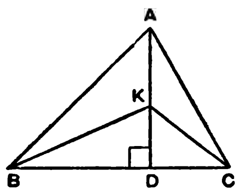


FIG. 549

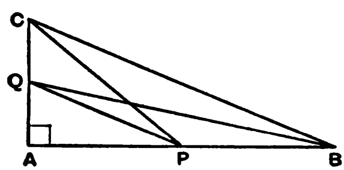


FIG. 550

9. In fig. 550, APB is perpendicular to AQC. Prove that $BQ^2 + CP^2 = BC^2 + PQ^2$.

[10] If, in fig. 550, where $\angle A = 1$ rt. $\angle$, P and Q are the mid-points of AB and AC, prove that $BQ^2 + CP^2 = 5PQ^2$.

[11] The diagonals of the quadrilateral ABCD cut at right angles. Prove that $AB^2 + CD^2 = AD^2 + BC^2$.

[12] ABCD is a rhombus. Prove that $AC^2 + BD^2 = 4AB^2$.

13. Given two squares, show how to construct a square whose area is equal to (i) the sum of the areas of the given squares, (ii) the difference of the areas of the given squares.

14. P is a point inside a rectangle ABCD. Prove that $PA^2 + PC^2 = PB^2 + PD^2$. [Drop perpendiculars from P to the sides of ABCD.] Is the same result true if P lies outside ABCD?

[15] P is a point on the base BC of the equilateral triangle ABC such that $BP = \frac{1}{3}BC$. Prove that $9AP^2 = 7AB^2$. [Draw the altitude AD; let $BC = 6x$ units.]

[16] In $\triangle$s ABC, PQR, $\angle B = \angle Q = 1$ rt. $\angle$. If $PQ = AB + BC$ and $QR = AB - BC$, prove that PR is equal to the diagonal of the square on AC.

17. In $\triangle ABC$, $\angle B$ is a right angle, X is the mid-point of BC and XN is the perpendicular from X to AC, prove that $AN^2 - NC^2 = AB^2$. [Join AX.]

18. ABCD is a quadrilateral in which $\angle B = \angle D = 1$ rt. $\angle$; AH, CK are the perpendiculars from A, C to BD. Prove that $BH^2 + BK^2 = DH^2 + DK^2$.

[19] ABCD is a quadrilateral in which $\angle ACB = \angle ADB = 1$ rt. $\angle$; AP, BQ are the perpendiculars from A, B to CD. Prove that $CP^2 + CQ^2 = DP^2 + DQ^2$.

20. In fig. 546, p. 282, $\angle BAC = 1$ rt. $\angle$ and $AB = 2AC$, prove that $BX = 4XC$. [What do you know about the areas of rect. BY, rect. CY?]

[21] In fig. 546, p. 282, $\angle BAC = 1$ rt. $\angle$ and $BX = 2\frac{1}{4}XC$, prove that $AB = 1\frac{1}{2}AC$.

22. In $\triangle ABC$, $AB = AC = 2BC$. If BF is an altitude, prove that $CF = \frac{1}{4}CB$. [In fig. 529, p. 270, what do you know about rect. FM?]

23. In fig. 547, p. 284, $\angle BAC = 1$ rt. $\angle$ and AX is perpendicular to BC. Prove that $AX^2 = BX \cdot XC$. [Let $AX = h$ units; use the small letters in the diagram and apply Pythagoras to each of the three right-angled triangles.]

Can you also prove this result trigonometrically?

24. The base BC of $\triangle ABC$ is produced to any point P. If $AB = AC$, prove that $AP^2 - AB^2 = PB \cdot PC$. [Draw the altitude AD; let $BC = 2x$ units, $CP = y$ units.]

*25. ABC is a straight line; ABXY, BCPQ are squares on the same side of AC. Prove that $PX^2 + CY^2 = 3(AB^2 + BC^2)$.

*26. O is any point inside $\triangle ABC$; OP, OQ, OR are the perpendiculars from O to BC, CA, AB respectively. Prove that $BP^2 + CQ^2 + AR^2 = PC^2 + QA^2 + RB^2$.

*27. If, in fig. 529, p. 270, which represents the squares on the sides of any triangle ABC, T is any point inside $\triangle$ ABC, prove that $TQ^2 + TM^2 + TK^2 = TP^2 + TN^2 + TH^2$.

*28. Given a line AB, construct, when possible, a point P in AB such that the sum of the squares on AP and PB is equal to the area of a given square. When is it impossible? [Use the fact stated in No. 5.]

*29. Given a line AB, construct a point P in AB such that $AP^2 = 2PB^2$. [Draw $\triangle$ AQB so that $\angle QAB = 45°$, $\angle QBA = 22\frac{1}{2}°$, and draw the perpendicular bisector of BQ.]

*30. Prove that, if A is any point on the circle whose diameter is BC, then $\angle BAC$ is a right angle. Use this fact and fig. 530, p. 270, to construct a square equal in area to a given rectangle.

## REVISION PAPERS 35–42 (Theorems 1–39)

[*Including areas of parallelograms and triangles.*]

### 35

1. In $\triangle$ ABC, AB = AC and D is a point on AC such that BD = BC. If $\angle CBD = x°$, find $\angle DBA$ in terms of $x$.

2. ABC is an equilateral triangle; P, Q are points on BC, CA respectively, such that BP = CQ; AP cuts BQ at R. Prove that (i) AP = BQ; (ii) $\angle BRA = 120°$.

3. Construct a parallelogram ABCD such that $\angle B = 60°$, BC = 3 in., and area of ABCD is 6·9 sq. in. Explain your method and measure AB.

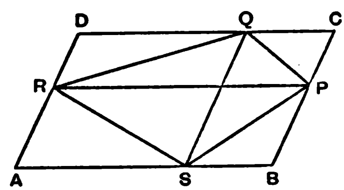


FIG. 551

4. In fig. 551, ABCD is a parallelogram, and PR, QS are parallel to AB, AD. Prove that area ABCD = twice area PQRS.

### 36

1. The angles of a triangle are $x°$, $x° + y°$, $x° + 2y°$, and one of the exterior angles is $3x° - y°$. Find the angles of the triangle. [Three sets of answers.]

2. ABCDE is a pentagon in which AB = AE, BC = ED, ∠ABC = ∠AED. Prove that ∠BCD = ∠EDC. [Join AC, AD.]

3. The vertices of a triangle are the points whose co-ordinates are (2, 0), (0, 5), (3, 7), and the unit on each axis is 1 inch. Calculate the area of the triangle.

4. The diagonals of the quadrilateral ABCD cut at K. If △AKD = △BKC, prove that the angles of △AKB are equal to the angles of △CKD.

## 37

1. The base BC of △ABC is produced to D; DA is joined and produced to E. If AB = AC = CD, prove that ∠BAE = 3∠BDA.

2. H is the mid-point of the side CD of the parallelogram ABCD; AH, BH produced meet BC, AD produced at X, Y respectively. Prove that (i) AH = HX; (ii) XY is parallel to AB.

3. Draw a parallelogram ABCD in which AB = 3·6 in., BC = 2·4 in., ∠B = 42°. Construct △PAB equal in area to ABCD and such that ∠PBA = 65°. Measure PB.

4. The side BC of the parallelogram ABCD is produced to any point K. Prove that △ABK = quad. ACKD.

## 38

1. In fig. 552, ∠PQC = ∠RQA, ∠QRC = ∠SRB, and CQA, CRB, PRS are straight lines. Prove that ∠QPR = 2∠ACB.

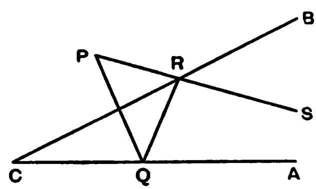


FIG. 552

2. K is any point inside an equilateral triangle ABC; BKD, CKE are equilateral triangles on the same sides of the bases BK, CK as A is. Prove that (i) △ADB ≡ △CKB; (ii) AD = KE; (iii) ADKE is a parallelogram.

3. Draw △ABC such that AB = 5 cm., BC = 3 cm., ∠B = 52°. Construct one position of a point P such that PA = PB and the area of △PBC equals 4·5 sq. cm. Explain your method shortly.

4. The diagonals of the quadrilateral ABCD cut at K. If △AKB = △BKC = △CKD, prove that ABCD is a parallelogram.

## 39

1. P is a point on the side AB of $\triangle$ ABC such that AP = PC = CB. If CP bisects $\angle$ ACB, calculate $\angle$ BAC.

2. BDEC is a square on the side BC of $\triangle$ ABC outside the triangle. Lines through B, C parallel to AD, AE meet at P. Prove that (i) PA = BC; (ii) PA is perpendicular to BC.

3. M is the mid-point of the side BC of $\triangle$ ABC; MP, MQ are the perpendiculars from M to AB, AC respectively. If AB = 8 in., AC = 12 in., MP = 6 in., calculate the length of MQ.

4. ABCD is a parallelogram; Q is any point on AB produced. Prove that $\triangle$ AQD − $\triangle$ BQC = $\triangle$ DQC. [Join BD.]

## 40

1. In $\triangle$ ABC, AB = AC and $\angle$ B = 51°. K is a point on AC produced such that $\angle$ AKB = 25°. Which is the longer, (i) BC or KC, (ii) BK or AK?

2. In $\triangle$ ABC, $\angle$ BAC = 90°; BDEC is a square outside $\triangle$ ABC; DX is the perpendicular from D to AC. Prove that DX = AB + AC. [Draw the perpendicular BY from B to DX.]

3. Draw a convex quadrilateral ABCD in which AB = 2 in., BC = 2·5 in., CD = DA = 1·7 in., $\angle$ B = 70°. Construct a point X on BC produced such that $\triangle$ ABX = quad. ABCD. Find the area of ABCD.

4. P, Q, R are points on the sides BC, CA, AB respectively of $\triangle$ ABC such that PQ is parallel to AB, and QR is parallel to BC. Prove that $\triangle$ ABP = $\triangle$ ACR.

## 41*

1. The sides AB, AC of $\triangle$ ABC are produced to D, E; AH, AK are lines parallel to the bisectors of $\angle$ BCE, $\angle$ CBD meeting BC in H, K. Prove (i) AC = CH; (ii) AB + AC − BC = HK.

2. In fig. 553, KE, KF are the bisectors of $\angle$ AEB, $\angle$ AFD. If $\angle$s DAB, DCB are supplementary, prove that

(i) $\angle$ CEF + $\angle$ CFE = $\angle$ EAF.

(ii) $\angle$ EKF is a right angle.

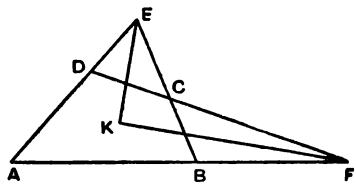



FIG. 553

3. AD, BE, CF are the altitudes of $\triangle$ ABC; AB $=39x$ in., BC $=44x$ in., CA $=17x$ in., AD $=30$ in. Calculate the lengths of BE, CF.

Use the formula for the area of a triangle on p. 248 to find the value of $x$.

4. The side AD of the parallelogram ABCD is produced to any point P; BP cuts CD at Q. Prove that $\triangle$QPC $=\triangle$QDA.

## 42*

1. ABCD is a square and ABK is an equilateral triangle outside ABCD; KD cuts AC at P. Find the size of $\angle$ADK and prove that (i) $\angle$BPC $=\angle$DPC $=60°$; (ii) $\triangle$PKB $\equiv\triangle$PCB.

2. ABC is a triangle; AP is the perpendicular from A to the bisector of $\angle$ABC; PQ is drawn parallel to BC to cut AB at Q. Prove that AQ $=$ QB $=$ QP.

[Produce AP to meet BC at R.]

3. AD, BC are the parallel sides of the trapezium ABCD; the diagonals AC, BD cut at K. If the areas of $\triangle$s AKB, BKC are 3 sq. in., 2 sq. in., respectively, calculate the areas of $\triangle$s CKD, AKD. Give reasons.

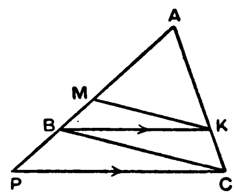


FIG. 554

4. In fig. 554, ABP is a straight line and M is the mid-point of AP. Prove that KM bisects the area of $\triangle$ABC. [Join PK.]

# REVISION PAPERS 43–50 (Theorems 1–41)

[*Including Pythagoras' Theorem.*]

## 43

1. ABC is an acute-angled triangle; BAHK, CAXY are squares outside $\triangle$ABC. If BH and CX when produced meet at P, prove that (i) A and P are on opposite sides of BC, (ii) $\angle$BPC $+\angle$BAC $=90°$.

2. (i) ABXY is a rectangle in which AB $=13$ in., AY $=6$ in.; C is a point on XY such that XC $=8{\cdot}5$ in. Find (i) the area of $\triangle$ABC, (ii) the length of AC, (iii) the length of the perpendicular from B to AC.

(ii) In $\triangle$ABC, $\angle$A $=90°$, $\angle$B $=30°$, BC $=6$ in.; calculate the area of $\triangle$ABC.

3. ABP, ABQ are triangles of equal area on opposite sides of AB. Prove that AB (produced if necessary) bisects PQ.

4. AD is an altitude of the acute-angled triangle ABC. If $BD = 2DC$, prove that $AB^2 = AC^2 + 3CD^2$.

## 44

1. In $\triangle ABC$, $\angle A = 120°$, P, Q are points on BC such that $BP = BA$ and $CQ = CA$. Prove that $\angle PAQ = 30°$.

2. APQR is a rectangle in which $AP = 4$ in., $AR = 11$ in. B, C are points on QR, QP respectively such that $RB = 3$ in., $QC = 8$ in. Calculate (i) the area of $\triangle ABC$; (ii) the length of the line joining the mid-points of BA and BC.

3. ABCD is a parallelogram; P is any point on BD. Prove that $\triangle PAB = \triangle PBC$. [Let AC cut BD at K.]

4. ABC is an equilateral triangle; BC is bisected at D and produced to E so that $CE = DC$. Prove that $AE^2 = 7EC^2$. [Let $BC = 2x$ units.]

## 45

1. (i) In $\triangle ABC$, $AB = 5$ in., $BC = 6$ in., $\angle ABC = 45°$. Calculate the area of $\triangle ABC$ and the length of AC.

(ii) ABCD is a quadrilateral in which $\angle A = \angle B = 90°$, $AB = 6$ in., $BC = 10$ in., $AD = 18$ in. Prove that BD bisects $\angle ADC$.

2. In $\triangle ABC$, $AB > AC$; CT is the perpendicular from C to the line bisecting $\angle BAC$; D is the mid-point of BC. Prove that (i) DT is parallel to BA; (ii) $DT = \frac{1}{2}(AB - AC)$. [Produce CT to meet AB at K.]

3. K is the mid-point of the diagonal BD of the quadrilateral ABCD. Prove that the difference between the areas of $\triangle ABC$, $\triangle ADC$ is equal to twice the area of $\triangle AKC$ [Let AC cut BD at N.]

4. In $\triangle ABC$, $\angle ACB$ is a right angle. If D is the mid-point of BC, prove that $AB^2 = AD^2 + 3BD^2$.

## 46

1. In $\triangle ABC$, $AB = AC$; P is any point on BC. If the perpendicular bisectors of PB, PC meet AB, AC respectively at X, Y, prove that AXPY is a parallelogram.

2. ABCD is a trapezium of area 36 sq. cm. in which AB is parallel to DC, and $AD = BC$. If $AB = 12$ cm., and $CD = 6$ cm., find the length of BC.

3. In fig. 555, $PQ = \frac{1}{3}BC$. Prove that quad. $AXRY = \frac{1}{3}\triangle ABC$.

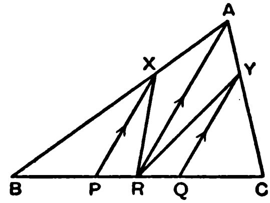


FIG. 555

4. In fig. 530, p. 270, which represents squares on the sides of a right-angled triangle, prove that

(i) $CK = BN$.
(ii) $CH^2 - BM^2 = AB^2 - AC^2$.

Prove these results are also true for fig. 529, p. 270.

## 47

1. $AD$ is an altitude of $\triangle ABC$ and $D$ lies between $B$ and $C$. If $AB = 10$ cm., $AC = 7{\cdot}5$ cm., $AD = 6$ cm., find the length of $BC$ and prove that $\angle BAC$ is a right angle.

2. $A$, $B$ are fixed points; $P$ is a variable point such that $\angle APB$ is obtuse; the perpendicular bisectors of $PA$, $PB$ cut $AB$ at $Q$, $R$. Prove that the perimeter of $\triangle PQR$ is constant.

3. $ABCD$ is a rhombus; $P$, $Q$ are points on $BC$, $CD$ respectively such that $BP = CQ$; $AP$ cuts $BQ$ at $K$.
Prove that $\triangle AKB =$ quad. $KPCQ$.

4. The base of a right pyramid, vertex $V$, is a square $ABCD$ of side 4 in., and each of the other four faces is an equilateral triangle. Prove that $\angle AVC$ is a right angle and calculate the height of the pyramid.

## 48

1. Given an equilateral triangle $ABC$, construct a point $P$ on $AB$ such that $PA$ is equal to the perpendicular $PN$ from $P$ to $BC$. Give reasons.

2. In $\triangle ABC$, $AB = 8$ cm., $AC = 9$ cm., and $D$ is a point on $BC$ such that $BD = \frac{1}{3}BC$. If the area of $\triangle ABC$ is 24 sq. cm., find the distances of $D$ from $AB$ and from $AC$.

3. In fig. 556, find the length of $AD$. If the perpendicular bisector of $AD$ cuts $AD$, $BC$ at $P$, $Q$, find (i) the length of $BQ$, (ii) the length of $PQ$. [Draw $DN$, $PK$ perp. to $BA$, $BC$.]

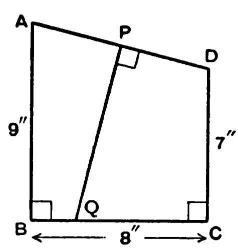


FIG. 556

4. AD is an altitude of $\triangle$ ABC; P, Q are points on AD produced such that DP = AB and DQ = AC. Prove that BQ = CP.

## 49*

1. (i) Find the area of a triangle whose sides are $(p^2 - q^2)$, $2\,pq$, $(p^2 + q^2)$ in. Give reasons.

(ii) ABCD is a parallelogram in which AB = 4 in., BC = 5 in., $\angle$ ABC = 60°. Calculate the area of ABCD, the distance of A from CD, and the length of BD.

2. In fig. 557, $\angle$ ABC = 90°, AK = BC, and E, F are the mid-points of AC, KB. Prove that $\angle$ AFE = 45°. [Draw CQ parallel to EF to cut AB produced at Q.]

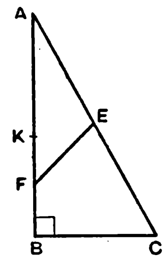


FIG. 557

3. ABCD is a quadrilateral in which

$$\angle \text{ABC} = \angle \text{ADC} = 90°.$$

Find a relation between the lengths of the sides of ABCD.

If AP, AQ are drawn parallel to CD, CB cutting CB, CD at P, Q respectively, prove that QA . AB = PA . AD.

[Use area-formulæ.]

4. ABCD is the base of a cuboid, and AP, BQ, CR, DS are the edges perpendicular to the base; AB = 4 in., AP = 3 in. If K is the mid-point of PQ and if AK = AD, find the length of AR.

## 50*

1. ABCD is a rhombus in which $\angle$ BAD = 120° and AB = 10 cm. Calculate the lengths of AC and BD.

2. ABCD is a square, side 3 cm., in a horizontal plane; AP is a vertical line 4 cm. long. Find the area of $\triangle$ PBC and the length of PC.

3. ABCD is a parallelogram; any line parallel to BA cuts BC, AC, AD at X, Y, Z respectively. Prove that $\triangle$ AXY = $\triangle$ DYZ. [Join CZ.]

4. AKB, CKD are two fixed lines such that $\angle$ BKD = 50°; P is a variable point such that its distance from AKB exceeds its distance from CKD by 1 cm. Draw a figure and show on it the *precise* locus of P. [Find two straight lines from which P is equidistant.]

# *PART II*

## (*Section* 2)

## THE CIRCLE

### Symmetrical Properties of the Circle

The chief definitions relating to the circle were given on p. 10, and many properties of the circle have already been discussed or given as examples, particularly in connection with loci.

It has been proved, see pp. 200, 201, that the locus of a point equidistant from two fixed points is the perpendicular bisector of the line joining the two fixed points.

This property involves two distinct theorems:

(1) If $PA = PB$, then P lies on the perpendicular bisector of AB.

(2) If Q lies on the perpendicular bisector of AB, then $QA = QB$.

If A and B are any two points on the circumference of a circle, centre O, then $OA = OB$ *radii*,

$\therefore$ O lies on the perpendicular bisector of AB.

In words,

**The centre of a circle lies on the perpendicular bisector of any chord of the circle.**

This statement is substantially equivalent to the facts expressed in Theorems 42, 43; but independent proofs of these theorems are given on pp. 300, 301 to meet examination requirements.

By using both facts established in the locus theorem, it can be proved, see p. 304, that one and only one circle can be drawn through three given points, not in the same straight line. The size and position of a circle is therefore fixed completely if three points A, B, C on its circumference are given; and so we can speak of the circle ABC without ambiguity because there is only one circle which passes through A, B, C.

## Examples for Oral Discussion

1. If in fig. 558, $AB = 8$ cm. and the radius of the circle is 5 cm., find the distance $OH$ of the centre $O$ from $AB$.

(i) What is the length of $AH$? Give reasons.
(ii) Use Pythagoras to find the length of $OH$.

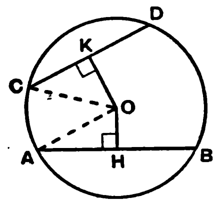



FIG. 558

2. If in fig. 558, the distance $OK$ of the centre $O$ from $CD$ is 6 in. and the radius is 7·5 in., find the length of $CD$.

3. **If two chords of a circle are unequal, prove that the greater is nearer the centre.**

Given in fig. 558 that $AB > CD$ and that $OH$, $OK$ are the perpendiculars from the centre $O$ to $AB$, $CD$, prove that $OH < OK$.

(i) Explain why $AH > CK$.
(ii) Explain why $AH^2 + HO^2 = CK^2 + KO^2$.

4. Calculate the circumradius, $r$ cm., of $\triangle ABC$, given that $AB = AC = 15$ cm., $BC = 18$ cm.

(i) Explain why the circumcentre $O$ lies on the line joining $A$ to the mid-point $N$ of $BC$.
(ii) Find the length of $AN$; then express the length of $ON$ in terms of $r$.
(iii) Use $\triangle ONB$ to find $r$.

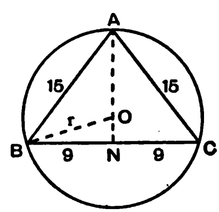



FIG. 559

# NUMERICAL EXAMPLES

## EXERCISE 56

[*Give answers which are approximate, correct to* 3 *figures.*]

**1.** A chord of length 10 cm. is at a distance of 12 cm. from the centre of the circle. Find the radius.

**2.** A chord of a circle of radius 6 cm. is 8 cm. long. Find the distance of the chord from the centre.

[3] A chord of a circle of radius 7 cm. is at a distance of 4 cm. from the centre. Find the length of the chord.

**4.** PQ is a variable chord of a given circle of radius 7·5 cm. If PQ = 9 cm., find the locus of the mid-point of PQ.

**5.** N is a point on the diameter AB of a circle APBQ; PNQ is the chord through N perpendicular to AB. If AN = 8 cm., NB = 2 cm., find the length of PQ.

[6] In a circle of radius 5 cm., there are two parallel chords of lengths 6 cm., 4 cm. Find the distance between the chords. [Two answers.]

[7] Two concentric circles are of radii 7 in., 4 in.; a line PQRS cuts one circle at P, S and the other at Q, R. If QR = 6 in., find the length of PS.

**8.** A chord of a circle is 10 cm. long and is 4 cm. from the centre. Find the length of a chord which is 3 cm. from the centre.

**9.** APB, CPD are two chords of a circle, centre O, radius 7 in., which intersect at right angles. If AB = 6 in., CD = 10 in., find the length of OP.

[10] The length of the common chord of two equal intersecting circles is 10 cm., and the distance between the two centres is 6 cm. Find the radius of each circle.

***11.** AB, CD are two chords of a circle at distances $d$ in., $7d$ in. from the centre. If AB = 2CD, find the length of CD in terms of $d$.

**12.** A hemispherical bowl, internal diameter 12 in., is partly full of water. If the depth of water below the centre is 4 in., find the diameter of the circular water-surface.

13. If fig. 560 represents a section of a circular cone, vertex V, by a plane through its axis VN, and the section of the sphere of radius 5 in. in which the cone is inscribed, and if the base-radius of the cone is 3 in., find the height VN of the cone.

14. In fig. 560, if $VA = VB = 10$ cm. and if the diameter of the circle VAB is 12 cm., find the lengths of VN and AB. [Let $ON = h$ units, $AN = d$ units, where O is the centre.]

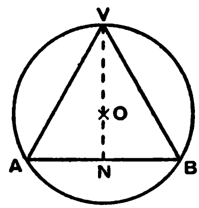



FIG. 560

[15] In fig. 560, if $VA = VB = 13$ in. and $AB = 10$ in., find the radius of the circle VAB. [Use the right-angled triangle ONA where O is the centre; let the radius $= r$ in.]

[16] If, in fig. 560, VAB is an equilateral triangle of side 6 cm., find the radius of the circle VAB. [Note that if O is the centre, $\triangle$ AON is "half an equilateral triangle."]

17. The perpendicular bisector ND of a chord AB of a circle, centre O, cuts AB at N and the circle at D, see fig. 561. If $AB = 6$ in., $ND = 1$ in., find the radius of the circle.

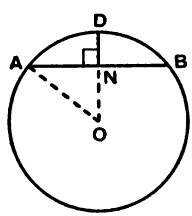



FIG. 561

*18. If, in fig. 561, $AN = NB = l$ in. and $ND = h$ in., prove that the diameter of the circle ADB is $\dfrac{l^2 + h^2}{h}$ in.

*19. In fig. 562, AB, CD are parallel chords of a circle, 3 in. apart. If AB = 4 in., CD = 10 in., find the radius of the circle. [Take the centre O; join OA, OC; let ON = $x$ in., OC = $r$ in.]

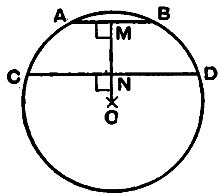



FIG. 562

*20. A crescent is formed of two circular arcs ACB, ADB, of equal radius, centres E, F, see fig. 563; the perpendicular bisector of AB cuts the crescent at C, D; CD = 5 cm., AB = 12 cm. Prove that EF = CD and find the radius of the arcs.

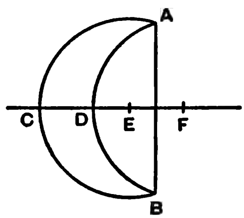



FIG. 563

*21. Two spheres, radii 6 in., 8 in., have their centres 10 in. apart. Find the radius of the circle in which the spheres cut one another, and the distances of the plane of this circle from the centres of the spheres.

*22. A thin rectangular plate, 3 in. by 4 in., rests horizontally in a hemispherical bowl, 13 in. in diameter. Find the height of the centre of the plate above the lowest point of the bowl.

*23. O is the centre of a *horizontal* circle, diameter 3 in., and OA is a horizontal line of length $2\frac{1}{2}$ in.; P is a point 2 in. vertically above A. If C is the centre of the sphere whose surface passes through the circle and the point P, find the length of CO and the radius of the sphere.

## THEOREM 42

**The straight line which joins the centre of a circle to the middle point of a chord, which is not a diameter, is perpendicular to the chord.**

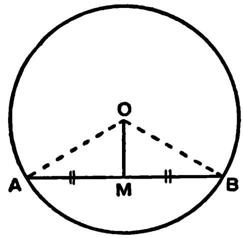



FIG. 564

**Given** a circle, centre O, and a chord AB whose mid-point is M, where M and O are different points.

**To prove that** $\angle$OMA is a right angle.

**Construction.** Join OA, OB.

**Proof.** In $\triangle$s OMA, OMB,

OA = OB *radii,*
AM = BM *given,*
OM = OM,

$\therefore$ $\triangle$s $\begin{matrix}\text{OMA}\\\text{OMB}\end{matrix}$ are congruent SSS.

$\therefore$ $\angle$OMA = $\angle$OMB.

But these are adjacent angles on a straight line,

$\therefore$ $\angle$OMA is a right angle.

**Corollary. The perpendicular bisector of a chord of a circle passes through the centre of the circle.**

If the chord is not a diameter, the line joining the centre of the circle to the mid-point of the chord is the perpendicular bisector of the chord.

If the chord is a diameter, its mid-point is the centre of the circle.

## THEOREM 43

**The straight line drawn from the centre of a circle perpendicular to a chord bisects the chord.**

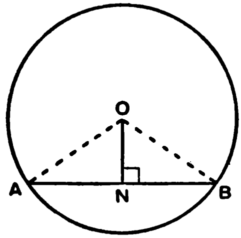



FIG. 565

**Given** a circle, centre O, and the perpendicular ON from O to a chord AB.

**To prove that** AN = NB.

**Construction.** Join OA, OB.

**Proof.** In △s ONA, ONB,

OA = OB *radii,*
ON = ON
∠ONA = ∠ONB *rt. ∠s, given,*

∴ △s $\begin{matrix} ONA \\ ONB \end{matrix}$ are congruent RHS.

∴ AN = BN.

**Corollary. The locus of the mid-points of parallel chords of a circle is a diameter of the circle.**

Since the chords are parallel, the diameter perpendicular to one of the chords is perpendicular to each of the others and therefore bisects it.

Conversely, any point on this diameter is the mid-point of the chord through this point perpendicular to the diameter.

## THEOREM 44

**If two chords of a circle are equal, they are equidistant from the centre.**

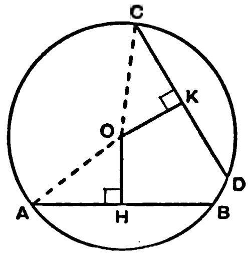



FIG. 566

**Given** a circle, centre O, and two equal chords AB, CD.

**To prove that** the perpendiculars OH, OK from O to AB, CD are equal.

**Construction.** Join OA, OC.

**Proof.** Since the perpendicular from the centre of a circle to a chord bisects the chord,

$$AH = \tfrac{1}{2}AB \text{ and } CK = \tfrac{1}{2}CD;$$

$$\text{but } AB = CD \qquad \textit{given,}$$

$$\therefore\ AH = CK.$$

In $\triangle$s OHA, OKC,

$$OA = OC \qquad \textit{radii,}$$

$$AH = CK \qquad \textit{proved,}$$

$$\angle OHA = \angle OKC \qquad \textit{rt. } \angle s, \textit{ given,}$$

$$\therefore\ \triangle s\ \begin{matrix} OHA \\ OKC \end{matrix} \text{ are congruent} \qquad \text{RHS.}$$

$$\therefore\ OH = OK.$$

**Corollary. In equal circles, equal chords are equidistant from the centres.**

The proof is similar.

### THEOREM 45

**If two chords of a circle are equidistant from the centre, their lengths are equal.**

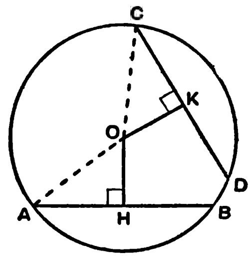



FIG. 567

**Given** a circle, centre O, and two chords AB, CD such that the perpendiculars OH, OK from O to AB, CD are equal.

**To prove that** AB = CD.

**Construction.** Join OA, OC.

**Proof.** In $\triangle$s OHA, OKC,

$$OH = OK \qquad \textit{given},$$
$$OA = OC \qquad \textit{radii},$$
$$\angle OHA = \angle OKC \qquad \textit{rt. } \angle s\textit{, given},$$

$$\therefore \triangle s \begin{matrix} OHA \\ OKC \end{matrix} \text{ are congruent} \qquad \text{RHS.}$$

$$\therefore AH = CK.$$

Since the perpendicular from the centre of a circle to a chord bisects the chord,

$$AH = \tfrac{1}{2}AB \text{ and } CK = \tfrac{1}{2}CD,$$
$$\therefore AB = CD.$$

**Corollary. In equal circles, chords which are equidistant from the centres are equal.**

The proof is similar.

## THEOREM 46

**There is one circle and only one circle which passes through three given points not in the same straight line.**

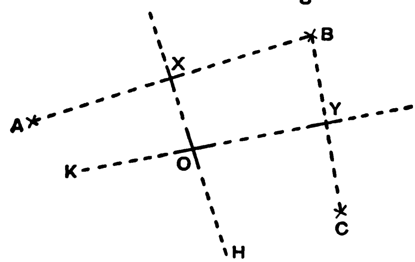



FIG. 568

**Given** three points A, B, C not in the same straight line.

**To prove that** one, and only one, circle can be drawn to pass through A, B, C.

**Construction.** Join AB, BC.

Draw the perpendicular bisectors XH, YK of AB, BC.

**Proof.** Since AB and BC are not in the same straight line, the perpendiculars XH, YK are not parallel and therefore intersect at a point O.

Since XH is the perpendicular bisector of AB, it is the locus of points equidistant from A and B.

Similarly, YK is the locus of points equidistant from B and C.

∴ the point of intersection O of XH, YK is equidistant from A, B, C.

But any point which is equidistant from A, B, C must lie on XH and on YK,

∴ O is the only point which is equidistant from A, B, C.

∴ the circle, centre O, radius OA, passes through A, B, C, and there is no other circle which passes through A, B, C.

**Corollary. Two distinct circles cannot intersect in more than two points.**

If two distinct circles cut at 3 points A, B, C, there would be more than one circle passing through A, B, C.

**The Circumcircle of a Triangle.** Given a triangle ABC, the circle ABC is called the **circumcircle** of $\triangle$ABC, its centre is called the **circumcentre** and its radius is called the **circumradius**, see p. 202. The method for constructing the circumcentre and circumcircle is given in Theorem 46.

To construct the centre of a circle an arc of which is given, it is sufficient to take any three points A, B, C on the arc and construct the circumcentre of $\triangle$ABC, because only one circle can be drawn through three given points.

**Symmetry about an Axis.** If one part of a figure can be made to coincide with the rest of the figure by folding it about a straight line AB, the figure is said to be **symmetrical** about AB, and the straight line AB is called an **axis of symmetry** of the figure. It then follows that the length of any line or the size of any angle in one-half of the figure is equal to the length of the corresponding line or the size of the corresponding angle in the other half of the figure.

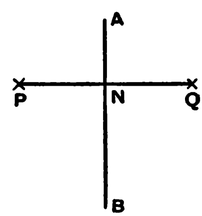


FIG. 569

If, in fig. 569, P coincides with Q when the figure is folded about AB, and if PQ cuts AB at N, $\angle$PNA coincides with $\angle$QNA, and therefore each is a right angle; also PN = QN.

Therefore, if P and Q are corresponding points for an axis of symmetry AB, the perpendicular bisector of PQ is AB. Conversely, if AB is the perpendicular bisector of PQ, then P and Q are corresponding points for the axis of symmetry AB; and we say that Q is the **image** of P in AB, and that P is the image of Q in AB.

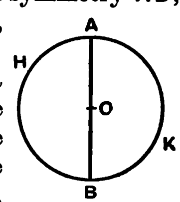


FIG. 570

If, in fig. 570, O is the centre and AB is a diameter of the circle AHBK, and if the figure is folded about AB, all points on the semicircle AHB in their new positions are still at a distance from O equal to the radius, and therefore coincide with points on the semicircle AKB. Therefore

**A circle is symmetrical about any diameter.**

Many properties of the circle which are proved by congruent triangles can also be proved by using this fact:

*e.g.* Theorems 42, 43, and the property that the common chord of two intersecting circles is bisected at right angles by the line joining the centres.

## EXERCISE 57

1. A straight line PQRS cuts two concentric circles at P, S and Q, R. Prove that PQ = RS. [Draw the perpendicular from the centre of the circles to PS; no other construction.]

[2] If AB and CD are equal chords of a circle, centre O, prove that $\angle$AOB = $\angle$COD.

3. M is the mid-point of a chord AB of a circle, centre O; MO is produced to any point T. Prove that TM bisects $\angle$ATB.

[4] O is the centre of the circle PQRS. If PQ = RS, prove that PR = QS. [Join O to P, Q, R, S.]

5. Given a point K inside a given circle, show how to construct the chord AKB such that AK = KB.

6. Two circles, centres A, B, intersect at X, Y. Prove that AB bisects XY at right angles.

[7] Two circles intersect at X, Y; a line PQRS parallel to XY cuts one circle at P, S and the other circle at Q, R. Prove that PQ = RS. [Use No. 6.]

8. Two circles, centres A, B, intersect at C, D; PCQ is a line parallel to AB cutting the circles at P, Q. Prove that PQ = 2AB

[9] APB, CPD are intersecting chords of a circle, centre O. If OP bisects $\angle$APC, prove that AB = CD. [Draw OH, OK perpendicular to AB, CD.]

10. AB and CD are two equal chords of a circle; M, N are their mid-points. Prove that MN makes equal angles with AB and CD.

[11] PQ is a variable chord of given length of a given circle. Find the locus of the mid-point of PQ.

12. A chord AB of a circle, centre O, is produced to P so that $BP = 2AB$. Prove that $OP^2 = OA^2 + 6AB^2$. [Draw ON perpendicular to AB; let $ON = d$ units, $AN = l$ units.]

[13] A, B, C are three points on a circle such that $\angle BAC$ is a right angle. Prove that the mid-point of BC is the centre of the circle. [Draw the perpendicular bisectors of AB, AC.]

14. Two chords of a circle bisect each other at K. Prove that K is the centre of the circle. [If possible, *join* K to the centre.]

[15] ABCD is a given quadrilateral. Show how to construct two concentric circles, one of which passes through A, B and the other through C, D What can you say about ABCD (i) if there is more than one solution, (ii) if two *distinct* circles cannot be drawn, (iii) if there is no solution?

16. The diagonals of the quadrilateral ABCD meet at O; circles are drawn through A, O, B; B, O, C; C, O, D; D, O, A. Prove that their four centres are the vertices of a parallelogram.

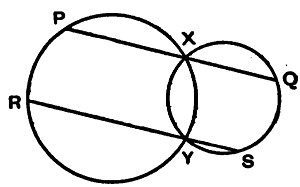


FIG. 571

17. In fig. 571, if PXQ is parallel to RYS, prove that $PQ = RS$. [Draw perpendiculars from the centres to PQ, RS.]

[18] If A, B are the centres of the circles in fig. 571, prove that PQ is not greater than 2AB.

19. A variable circle cuts off equal intercepts from two given lines AB, AC, produced if necessary. Find the locus of its centre.

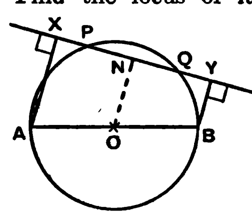


FIG. 572

[20] ABC is a given triangle. Show how to construct a circle to pass through B and C and to cut off equal intercepts from AB, AC, produced if necessary.

21. In fig. 572, AX, BY are the perpendiculars to a chord PQ from the ends A, B of a diameter. Prove that $XP = QY$.

Is the same result true if P, Q lie on opposite sides of AB? [Use the intercept theorem.]

[22] Two circles, centres A, B, cut at X, Y; M is the mid-point of AB. If the line through X perpendicular to MX cuts the circles again at P and Q, prove that $XP = XQ$. [Draw the perpendiculars from A, B to PQ and use the intercept theorem.]

23. In fig. 573, A, C, B are the centres of three unequal circles. If AC = CB, prove that PQ = RS. [PS is not parallel to AB.]

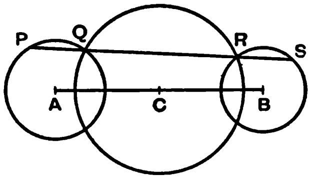


Fig. 573

*24. Two variable chords PAQ, RAS of a fixed circle intersect at right angles at a fixed point A. Prove that $PQ^2 + RS^2$ is constant.

*25. X, Y are the mid-points of the chords AB, CD of a circle, centre O; XN, YM are the perpendiculars from X, Y to CD, AB respectively. If XN cuts YM at P, prove that OP and XY bisect each other.

*26. P is any point on a diameter AB of a circle; QPR is a chord such that $\angle APQ = 45°$. Prove that $AB^2 = 2PQ^2 + 2PR^2$.

## Angle Properties of a Circle

**Definitions.** (1) Any part of the circumference of a circle is called an **arc** of the circle: it is called a **minor** arc if it is less than half the circumference and a **major** arc if it is greater than half the circumference.

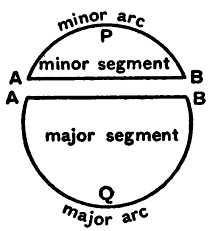


Fig. 574

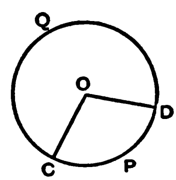


Fig. 575

(2) The plane figure bounded by a chord of a circle and either of the arcs it cuts off is called a **segment** of the circle: it is called a **minor** segment if it is less than a semicircle and a **major** segment if it is greater than a semicircle. In fig. 574, the figure bounded by the chord AB and the minor arc APB is a minor segment; the figure bounded by the chord AB and the major arc AQB is a major segment.

(3) The figure bounded by two radii of a circle and either of the arcs they cut off is called a **sector** of a circle.

In fig. 575, the figure bounded by the radii OC, OD and the arc CPD is a sector; so also is the figure bounded by the radii OC, OD and the arc CQD.

(4) Any number of points are said to be **concyclic** if a circle can be drawn to pass through all of them.

If the four vertices of a quadrilateral are concyclic, the quadrilateral is called a **cyclic quadrilateral.**

## Examples for Oral Discussion

**1.** In fig. 576, O is the centre of the circle and PON is a straight line.

(i) Name two equal angles in the figure.
(ii) If $\angle$APN $=24°$, find $\angle$AON.
(iii) If $\angle$AON $=52°$, find $\angle$APN.
(iv) If $\angle$APN $=x°$, find $\angle$AON.

*Give reasons for each answer.*

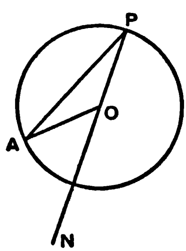



Fig. 576

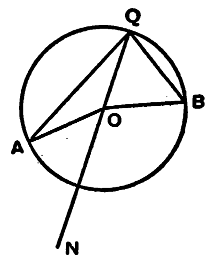



Fig. 577

**2.** In fig. 577, O is the centre of the circle and QON is a straight line.

(i) Name two pairs of equal angles in the figure.
(ii) If $\angle$AQN $=20°$ and $\angle$BQN $=50°$, find $\angle$AON and $\angle$BON. Find also $\angle$AQB and $\angle$AOB.
(iii) If $\angle$AQN $=x°$ and $\angle$BQN $=y°$, find $\angle$AQB and $\angle$AOB in terms of $x$ and $y$.
(iv) What can you say about $\angle$AQB if AOB is a straight line?

**3.** In fig. 578, O is the centre of the circle and RON is a straight line.

(i) Name two pairs of equal angles in the figure.

(ii) If $\angle$ARN $=30°$ and $\angle$BRN $=70°$, find $\angle$AON and $\angle$BON. Find also $\angle$ARB and $\angle$AOB.

(iii) If $\angle$ARN $=x°$ and $\angle$BRN $=y°$, find $\angle$ARB and $\angle$AOB in terms of $x$ and $y$.

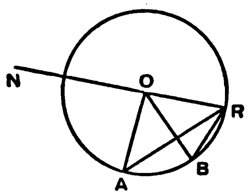



FIG. 578

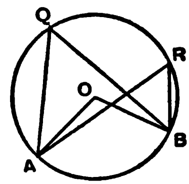



FIG. 579

**4.** In fig. 579, O is the centre of the circle AQRB.

Use the facts proved in 2 (iii) and 3 (iii) to state the connection between the sizes of $\angle$AOB, $\angle$AQB, $\angle$ARB. Also express the fact in the form of a general statement.

## Angles at the Centre and Circumference of a Circle

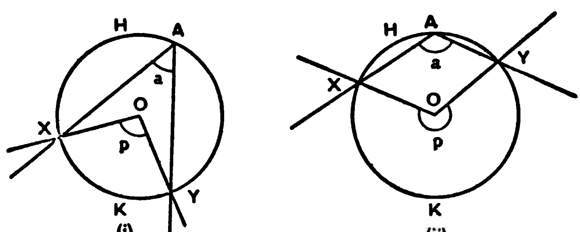



FIG. 580

In fig. 580 (i), (ii), $\angle p$ is an angle whose vertex is at the centre O of the circle and is called **the angle at the centre standing on the arc XKY**; $\angle a$ is an angle whose vertex A is a point on the circumference of the circle and is called **an angle at the circumference standing on the arc XKY.**

Thus in fig. 580, $\angle p$ and $\angle a$ are angles standing on the same arc, one at the centre and the other at the circumference.

It is necessary to use *three letters* to name the arc XKY on which the angles stand in order to distinguish it from the arc XHY. The angle at the centre standing on the arc XHY is $\angle q$, see fig. 581 (i), (ii), and an angle at the circumference standing on the arc XHY is $\angle b$.

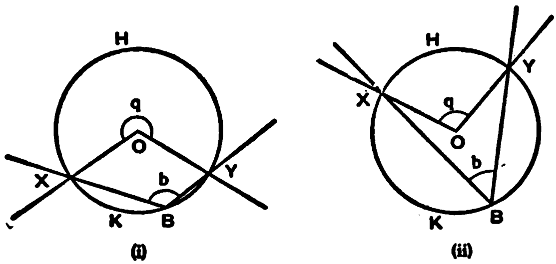



FIG. 581

We also say that the arc XHY subtends $\angle q$ at the centre and subtends $\angle b$ at the point B on the circumference.

It is important to learn how to see at a glance the arc on which an angle stands:

Look at the two points in which *the arms of the angle* cut the circumference.

The *arms* of $\angle q$ in fig. 581 cut the circumference in X and Y, and the arc which lies *inside* the angle $q$ is XHY, *not* XKY, and we therefore say that $\angle q$ stands on the arc XHY.

Similarly, the *arms* of $\angle b$ in fig. 581 cut the circumference in X and Y, and the arc which lies *inside* the angle $b$ is XHY, and we therefore say that $\angle b$ stands on the arc XHY.

## Angles in a Segment

Fig. 582 (i), (ii) show a circle XHYK; XHY is a segment of the circle bounded by the chord XY and the arc XHY; A, B, C are any points on the arc XHY.

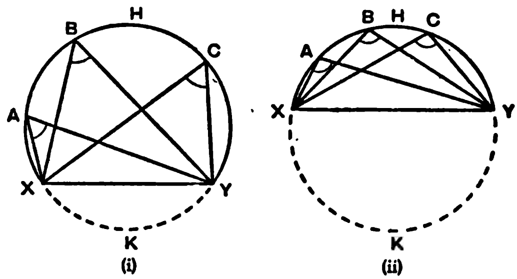



FIG. 582

We say that ∠XAY, ∠XBY, ∠XCY are angles **IN** the segment XHY.

The *vertex* of any angle *in* a segment is *a point on the arc* which bounds the segment, and the *arms* of the angle pass through the *ends of the chord* which bound the segment.

Be careful to notice that any angle in the segment XHY *stands* on the arc XKY, *not* on the arc XHY which bounds the segment.

### Example for Oral Work

Draw on the blackboard a *large* circle and mark (say) eight points A, B, C, D, E, F, G, H at irregular intervals on its circumference. Join each pair of points by a straight line, and discuss such questions as:

(i) On what arc do the angles BCE, HAB, etc., stand?

(ii) What angles stand on the arc BCD, EGA, etc.?

(iii) What angles stand on the same arc as ∠EAC, ∠HBD, etc.?

(iv) What angles stand on the chord EC, HD, etc.?

(v) Name two angles which stand on the chord AE but do not stand on the same arc.

(vi) What angles are in the segment ABD, HEB, etc.?

## THEOREM 47

**The angle which an arc of a circle subtends at the centre is double that which it subtends at any point on the remaining part of the circumference.**

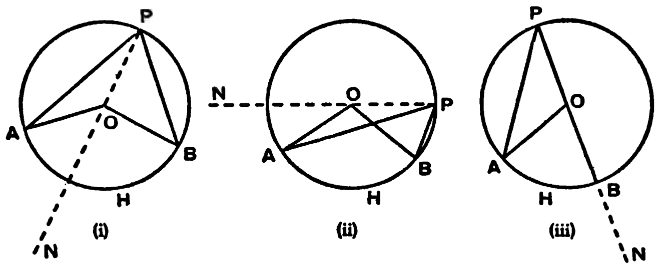



FIG. 583

**Given** a minor arc AHB of a circle, centre O, and a point P on the remaining part of the circumference.

**To prove that** $\angle AOB = 2\angle APB$.

**Construction.** Join PO and produce it to any point N.

**Proof.** $OA = OP$ *radii*,

$\therefore\ \angle OAP = \angle OPA$ *base* $\angle s$, *isos.* $\triangle$.

But $\angle NOA$ is an exterior angle of $\triangle AOP$,

$\therefore\ \angle NOA = \angle OAP + \angle OPA$,

$\therefore\ \angle NOA = 2\angle OPA$.

Similarly, $\angle NOB = 2\angle OPB$.

$\therefore$ adding in fig. 583 (i) and subtracting in fig. 583 (ii),

$\angle AOB = 2\angle APB$.

In fig. 583 (iii), where PO produced passes through B,

$\angle BOA = \angle NOA = 2\angle OPA$ *proved.*

In fig. 584, the arc AHB is a major arc and the angle AOB at the centre standing on the arc AHB is reflex. The proof in this case, and in the case where the arc AHB is a semicircle, is the same as for fig. 583 (i).

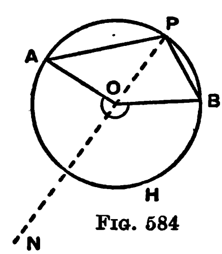



FIG. 584

## Examples for Oral Discussion

**1.** Fig. 585 represents a circle, centre O, and the small letters denote angles.

(i) If $p = 110°$, find $a$ and $b$.

(ii) If $b = 62°$, find $p$ and $a$.

(iii) If $a = 52°$, find $b$. Give reasons.

(iv) If $c = 125°$, find $q$ and $d$.

(v) If $q = 260°$, find $c$ and $d$.

(vi) If $a = 50°$, find $p$; then find $q$, $c$. How much is $a + c$?

(vii) If $p = 108°$, find $b$ and $d$. Give reasons. How much is $b + d$?

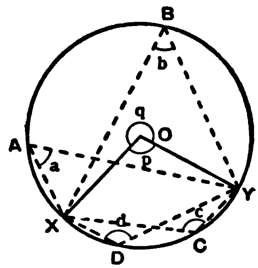


FIG. 585

(viii) If $a = 48°$, find $d$. (ix) If $c = 138°$, find $b$.

(x) If $b$ were 90°, what would $p$ be? How must the figure be drawn to give this result?

(xi) What points must be joined in the figure to form an angle double $\angle$ AXB?

(xii) Name angles in the figure which equal half $\angle$DOC.

**2.** In fig. 586, the angles p, q, r are angles in the major segment AKB of a circle, centre O. Prove that p = q = r.

(i) What do you know about $p$? Give reasons and complete the proof.

(ii) Draw a figure showing three angles $p$, $q$, $r$ in a *minor* segment AKB of a circle and explain why they are equal.

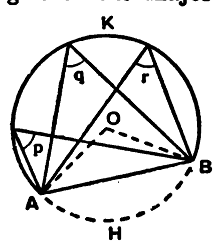


FIG. 586

(iii) Express the fact that has been proved in the form of a general statement.

(iv) If $p = 40°$, find the angles of $\triangle$AOB. Hence on a given line AB, 5 cm. long, construct a segment of a circle to contain an angle of 40°.

**3.** In fig. 587 (i), AKB is a major segment of a circle; in fig. 587 (ii), AKB is a semicircle; in fig. 587 (iii), AKB is a minor segment of a circle. Prove that

(i) the angle p is acute;
(ii) the angle q is a right angle;
(iii) the angle r is obtuse.

Express these facts in the form of general statements.

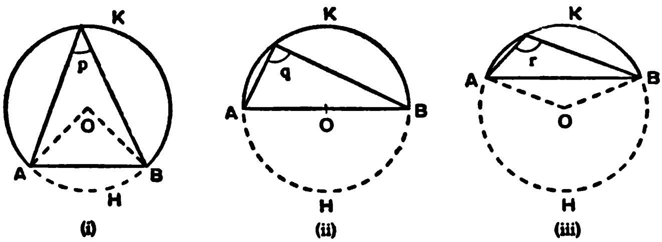



FIG. 587

**4.** (i) In fig. 588 (i), the angles b, d are the opposite angles of a cyclic quadrilateral ABCD. Prove that

$$b + d = 2 \text{ rt. } \angle s.$$

(ii) In fig. 588 (ii), p is the exterior angle of the cyclic quadrilateral formed by producing AD. Prove that

$$p = b.$$

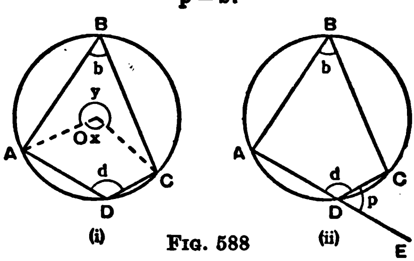



FIG. 588

(i) In fig. 588 (i), what do you know about *b*, about *d*?
(ii) In fig. 588 (ii), what do you know about *p* and *d*?
(iii) Express the two facts that have been proved in the form of general statements.

**5.** In fig. 589, PABQ, RDCS are straight lines cutting the circle ABCD. *Give short reasons* for each answer to the following:—

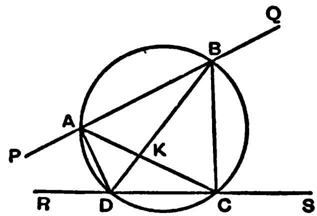



Fig. 589

(i) Which angle equals (*a*) $\angle$DAC, (*b*) $\angle$BDC?

(ii) Which angle equals (*a*) $\angle$QBC, (*b*) $\angle$BAD?

(iii) What follows if BD is a diameter?

(iv) If $\angle$ABD $=$ $\angle$CBD, name another pair of equal angles.

(v) If KC $=$ KD, name four equal angles.

**6.** Draw a large circle and mark four points A, B, C, D in order on its circumference. Use your figure for the following:

(i) Mark a point E on the circumference such that $\angle$AEC $=$ $\angle$ABC.

(ii) Mark a point F on the circumference such that $\angle$AFB, $\angle$ACB are supplementary.

(iii) Draw an angle $p$ in the minor segment cut off by BC.

(iv) Draw an angle $q$ in the major segment cut off by CD.

(v) Mark by the letters $r$, $s$ the two exterior angles of quadrilateral ABCD which are equal to $\angle$BCD.

**7.** Make a sketch of fig. 589 for each of the following examples and show the answers on it.

(i) Find all the angles you can if $\angle$PAD $=100°$, $\angle$DAC $=30°$.

(ii) Find all the angles you can if $\angle$ADB $=60°$, $\angle$DBC $=25°$.

(iii) Find all the angles you can if $\angle$RDB $=140°$, $\angle$QBD $=130°$.

(iv) Find the remaining angles if $\angle$RDA $=72°$, $\angle$ABD $=30°$, $\angle$ACB $=40°$.

(v) Find the remaining angles if $\angle$PAD $=115°$, $\angle$ADB $=50°$, $\angle$BKC $=85°$.

## NUMERICAL EXAMPLES

### EXERCISE 58

1. Two chords AB, CD of a circle intersect at right angles at a point inside the circle. If ∠BAC = 35°, find ∠ABD.

2. AB is a diameter of the circle ABCD. If ∠ADC = 127°, find ∠BAC.

[3] AC is a diameter of the circle ABCD. If ∠BDC = 25°, find ∠ACB.

4. ABCD is a cyclic quadrilateral. If ∠ADC = 70° and ∠ACD = 50°, find ∠CBD.

[5] Two chords AB, CD of a circle meet, when produced, at K. If ∠KAD = 31° and ∠AKC = 42°, find ∠KBC.

6. The diagonals of the cyclic quadrilateral ABCD cut at N. If ∠BAC = 42°, ∠BNC = 114° and ∠ADB = 33°, find ∠BCD.

[7] ABCD is a cyclic quadrilateral and EABF is a straight line. If ∠EAD = 82°, ∠FBC = 74° and ∠BDC = 50°, find the acute angle between AC and BD.

[8] ABC is a triangle inscribed in a circle, centre O. If ∠AOC = 130°, ∠BOC = 150° and if O lies inside △ABC, find ∠ACB.

9. In fig. 590, O is the centre of the circle; ABN is a straight line. Find ∠AOC.

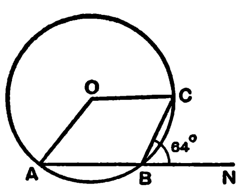



FIG. 590

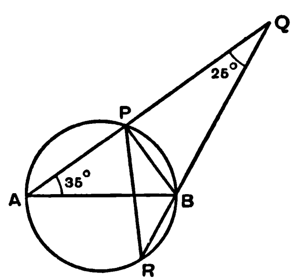



FIG. 591

10. In fig. 591, AB is a diameter of the circle APBR; APQ and RBQ are straight lines. Find ∠BPR.

[11] P, Q, R are points on a circle, centre O; $\angle$POQ $=54°$, $\angle$OQR $=36°$, and P, R are on opposite sides of OQ. Find $\angle$QPR and $\angle$PQR.

[12] P is a point on the minor arc AB of a circle, centre O. If $\angle$APB $=x°$ and $\angle$AOB $=y°$, find $x$ in terms of $y$.

13. ABCD is a convex quadrilateral in which AB = AC = AD. If $\angle$BAD $=140°$, find $\angle$BCD.

14. In fig. 592, O is the centre of the circle, OABC is a parallelogram, and BCP is a straight line. Find $\angle$OAP.

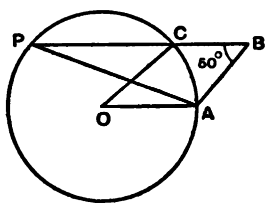



FIG. 592

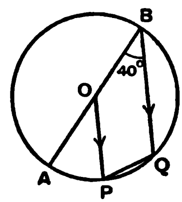



FIG. 593

15. In fig. 593, AB is a diameter and O is the centre of the circle. Find $\angle$OPQ.

[16] O is the centre of the circle ABC and $\angle$ABC $=30°$. If BC is parallel to OA, prove that AB is perpendicular to OC.

17. Two chords AB, DC of a circle, centre O, are produced to meet at E. If $\angle$AOB $=100°$, $\angle$EBC $=72°$, and $\angle$ECB $=84°$, find $\angle$COD.

[18] D is a point on the base BC of $\triangle$ABC; H, K are the centres of the circles ADB, ADC. If $\angle$AHD $=70°$ and $\angle$AKD $=80°$, find $\angle$BAC. [Two answers.]

19. Two circles APRB, AQSB intersect at A, B; PAQ, RBS are straight lines. If $\angle$QPR $=80°$ and $\angle$PRS $=70°$, find $\angle$PQS and $\angle$QSR. [Join AB.]

20. In fig. 594, AB is a diameter; ABRS and PQR are straight lines. Prove that PQ = QB.

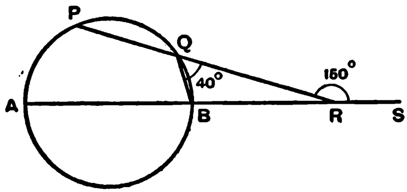



FIG. 594

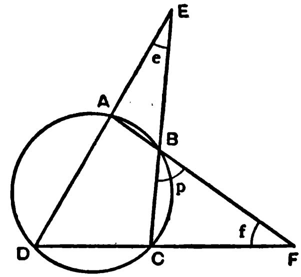



FIG. 595

*21. In fig. 595, if $e = 32°$ and $f = 40°$, find $p$.

*22. In fig. 595, prove that $e + f = 180° - 2p$.

*23. A chord QR and a diameter AB of a circle AQRB, when produced, meet at P. If ∠QPA = 28° and ∠QAR = 42°, find ∠QRA.

*24. In fig. 595, if AC cuts BD at K, and if $e = 40°$, $f = 20°$, ∠BKC = 100°, prove that ∠BAC = 2∠BCA.

*25. ABCDE is a pentagon inscribed in a circle. If ∠BDC = 20°, ∠CAD = 28°, ∠ABD = 70°, and if CD = DE, find the angles of the pentagon.

---

26. AB is a diameter of a circle APB, radius 5 cm. If AP = 6 cm., find the length of PB.

[27] Draw a circle of radius 4 cm. and mark a point A on the circumference. Inscribe a rectangle ABCD in the circle such that AB = 6 cm. Measure and calculate the length of BC.

28. Draw a triangle ABC such that AB = 10 cm., AC = 4 cm., ∠BAC = 90°. Construct a triangle APB equal in area to △ABC and such that ∠APB = 90°. Measure PA and PB.

[29] Construct a cyclic quadrilateral ABCD such that AB = 2·4 in., BC = 1·8 in., ∠ABC = 90° and AD = DC. Measure AD.

*30. Draw a rectangle 3 in. by 2 in. and construct without any arithmetical calculations a square equal in area to the rectangle. Measure the side of the square. [Use fig. 530, p. 270.]

## THEOREM 48

**Angles in the same segment of a circle are equal.**

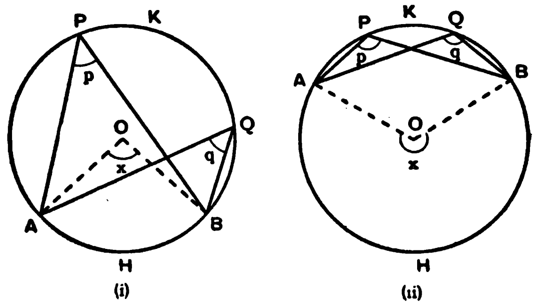



FIG. 596

**Given** a segment AKB of a circle AKBH and any two angles APB, AQB in the segment.

**To prove that** $\angle APB = \angle AQB$.

**Construction.** Let O be the centre of the circle. Join OA, OB.

**Proof.** With the notation in the figures, since $\angle$ at centre = twice $\angle$ at O^ce, standing on same arc,

$\angle x = 2\angle p$    *arc* AHB,

and $\angle x = 2\angle q$    *arc* AHB,

$\therefore \angle p = \angle q$.

**Corollary. The angle in a major segment of a circle is acute, and the angle in a minor segment of a circle is obtuse.**

$\angle x$ is less than 2 rt. $\angle$s if AHB is a minor arc, and is greater than 2 rt. $\angle$s if AHB is a major arc.

*For reference:* Same arc AHB

*or* same segment AKB.

## THEOREM 49

**The angle in a semicircle is a right angle.**

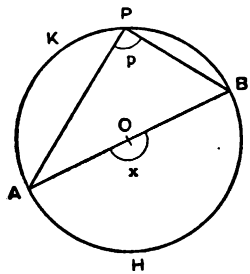



FIG. 597

**Given** a semicircle AKB of a circle AKBH and any angle APB in the semicircle.

**To prove that** $\angle$ APB = 1 right angle.

**Proof.** Since AKB is a semicircle, the centre O of the circle lies on AB.

With the notation in the figure,
since $\angle$ at centre = twice $\angle$ at O$^{ce}$, standing on same arc,

$$\angle x = 2\angle p \qquad \textit{arc } \mathsf{AHB}.$$

But since AOB is a straight line,

$$\angle x = 2 \text{ rt. } \angle\text{s}.$$

$$\therefore \angle p = 1 \text{ rt. } \angle.$$

*For reference:* $\angle$ in semicircle.

This fact was probably discovered by *Thales* (640–546 B.C.), one of the "Seven Wise Men." He introduced the study of geometry into Greece, and when in Egypt he showed King Amasis how to find the heights of the Pyramids by measuring their shadows and comparing the lengths with that of a shadow cast by a vertical pole.

## THEOREM 50

**(1) The opposite angles of a cyclic quadrilateral are supplementary.**

**(2) If one side of a cyclic quadrilateral is produced, the exterior angle so formed is equal to the interior opposite angle.**

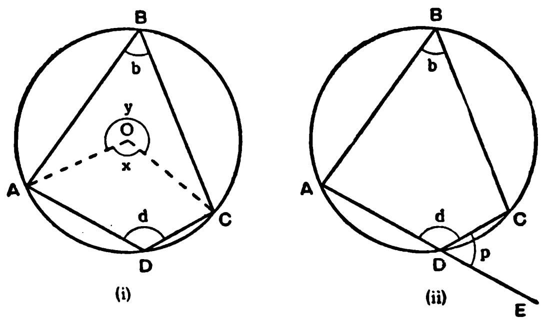



FIG. 598

**Given** a quadrilateral ABCD inscribed in a circle and one side AD produced to E.

**To prove that** (1) $\angle ABC + \angle ADC = 2$ right angles;

(2) $\angle CDE = \angle ABC$.

**Construction.** Let O be the centre of the circle. Join OA, OC.

**Proof.** (1) With the notation in fig. 598 (i),
since $\angle$ at centre = twice $\angle$ at O^ce^, standing on same arc,

$\angle x = 2\angle b$ *arc* ADC,

$\angle y = 2\angle d$ *arc* ABC,

$\therefore \angle x + \angle y = 2\angle b + 2\angle d.$

But $\angle x + \angle y = 4$ rt. $\angle$s *$\angle$s at a point,*

$\therefore 2\angle b + 2\angle d = 4$ rt. $\angle$s,

$\therefore \angle b + \angle d = 2$ rt. $\angle$s.

(2) With the notation in fig. 598 (ii),

| | | |
|---|---|---|
| | $\angle p + \angle d = 2$ rt. $\angle$s | *adj. $\angle$s on st. line,* |
| but | $\angle b + \angle d = 2$ rt. $\angle$s | *proved,* |
| | $\therefore \angle p + \angle d = \angle b + \angle d,$ | |
| | $\therefore \angle p = \angle b.$ | |

*For reference:* (1) opp. $\angle$s, cyclic quad.
(2) ext. $\angle$, cyclic quad.

NOTE. If Theorem 50 (2) is set by itself in an examination, the proof of Theorem 50 (1) must be included to secure full credit.

**Important Hints.** (i) In rider work, Theorem 50 (2) is more often of use than Theorem 50 (1).

(ii) In problems on intersecting circles, it is usually advisable to draw the common chord.

## EXERCISE 59

Nos. 1–4 refer to fig. 599, in which AY cuts BX at N.

1. If AB is parallel to XY, prove that $\angle$ANX $= 2\angle$ABX.

[2] If AB is parallel to XY, prove that NX = NY.

3. If YN = YB, prove that XN = XA.

[4] If XA and YB are produced to meet at K, and if AB is parallel to XY, prove that KX = KY.

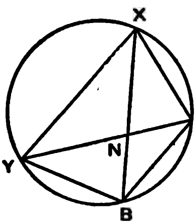


FIG. 599

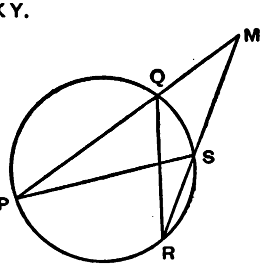


FIG. 600

Nos. 5–7 refer to fig. 600 in which PQM, RSM are straight lines.

5. Prove that $\angle$PSM $= \angle$RQM.

[6] If MQ = MS, prove that MP = MR.

7. If PS = SM, prove that $\angle$PQR $= 2\angle$QRS.

[8] ABCD is a cyclic quadrilateral. If AC bisects the angles BAD, BCD, prove that $\angle$ ABC is a right angle.

9. In fig. 601, PAQ, RAS are straight lines. Prove that $\angle$PXR = $\angle$QYS.

[10] In fig. 601, PAQ, RAS are straight lines. Join B to P, Q, R, S and prove that $\angle$PBQ = $\angle$RBS.

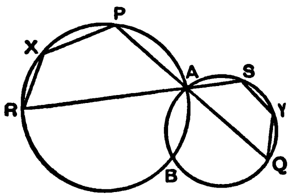



FIG. 601

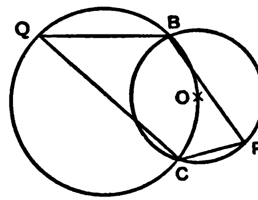



FIG. 602

11. In fig. 602, the centre O of circle BPC lies on circle BQC. Find a relation between $\angle$BPC and $\angle$BQC. [Join OB, OC.]

12. In fig. 603, CAL, CBM are straight lines. If CA is a diameter of the circle ABC, prove that $\angle$ALM is a right angle.

[13] Two circles intersect at A, B; AP, AQ are diameters of the circles. Prove PBQ is a straight line.

14. AP is a chord of a circle, centre O. If the circle on AO as diameter cuts AP at N, prove that AN = NP.

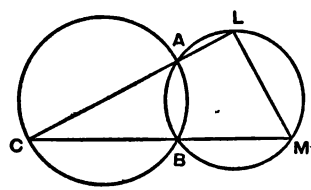



FIG. 603

15. AB is a diameter of the circle ADCB; the chord DC is produced to E. Prove that $\angle$ABD + $\angle$BCE = 90°.

[16] AB is a diameter of the circle APQRB. Prove that $\angle$APQ + $\angle$QRB = 270°. [Join AR.]

17. ABCDEF is a hexagon inscribed in a circle. Prove that $\angle$FAB + $\angle$BCD + $\angle$DEF is equal to 4 right angles.

[18] ABC is a triangle inscribed in a circle, centre O; N is the mid-point of BC. Prove that $\angle$BON, $\angle$BAC are equal or supplementary.

19. In fig. 604, PAQ, RBS are straight lines. Prove that PR is parallel to QS.

20. If, in fig. 604, PS cuts the circle PAB at K and cuts the circle QAB at H, prove that $\angle$PAH $= \angle$KBS.
[Use No. 19.]

[21] Two straight lines ABCD, PQRS are drawn to cut two circles ABQP, CDSR. If AP is parallel to CR, prove that BQ is parallel to DS.

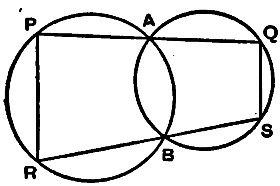



FIG. 604

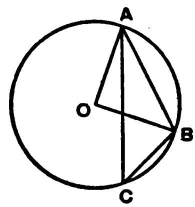



FIG. 605

22. In fig. 605, O is the centre of the circle ABC.
If $\angle$ACB $= \angle$OAB, prove that $\angle$AOB is a right angle.

[23] OP, OQ, OR are three equal lines. If $\angle$POQ $= 60°$, prove that $\angle$PRQ is either 30° or 150°.

[24] The side BA of the cyclic quadrilateral ABCD is produced to E. If AD bisects $\angle$CAE, prove that DB $=$ DC.

25. ABCD is a parallelogram. The circle through A, B, C cuts CD, produced if necessary, at E. Prove that AE $=$ AD.

[26] Two chords AB, CD of a circle, centre O, intersect at right angles at a point inside the circle.
Prove that $\angle$AOD $+ \angle$BOC equals two right angles.

27. Two chords AB, CD of a circle, centre O, intersect at a point N inside the circle. If $\angle$ANC is acute, prove that $\angle$AOC $+ \angle$BOD $= 2\angle$ANC.

[28] If O is the centre of the circle in fig. 600, p. 323, prove that $\angle$POR $- \angle$QOS $= 2\angle$PMR.

29. ABCP is a cyclic quadrilateral. Prove that a triangle whose sides are parallel to PA, PB, PC is equiangular to $\triangle$ABC.
[Draw HK $\parallel$ PC to cut PA, PB at H, K.]

[30] ABCD is a rectangle; any circle through A cuts AB, AC, AD at X, Y, Z. Prove that $\triangle$ XYZ is equiangular to $\triangle$ CBA.

31. In fig. 606, PQRS is a straight line. If BA bisects $\angle$ PAS, prove that BQ = BR.

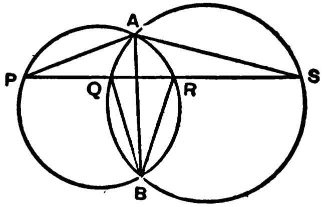



FIG. 606

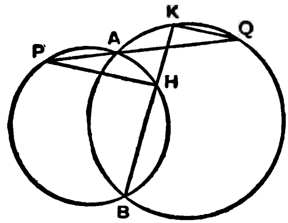



FIG. 607

32. In fig. 607, PAQ and BHK are straight lines. Prove that PH is parallel to KQ. [Join AB.]

33. D is any point on the side AB of $\triangle$ ABC; points E, F are taken on AC, BC respectively so that $\angle$ EDA = 60° = $\angle$ FDB; a circle is drawn through D, E, F and cuts AB again at G. Prove that $\triangle$ EFG is equilateral.

[34] P is any point on the minor arc AC of the circle ABC; BP cuts AC at Q. If $\triangle$ ABC is equilateral, prove that

$$\angle PAB = \angle AQB.$$

35. P is any point on the minor arc AC of the circle ABC; AP produced meets BC produced at Q. If AB = AC, prove that $\angle$ AQB = $\angle$ ABP.

[36] The bisectors of $\angle$ ABC, $\angle$ ACB of $\triangle$ ABC meet AC, AB at X, Y respectively and intersect at Z. If AXZY is a cyclic quadrilateral, prove that $\angle$ BAC = 60°.

37. K is a point inside $\triangle$ ABC; BK, CK produced meet AC, AB at E, F respectively. If AEKF and BFEC are cyclic quadrilaterals, prove that BE and CF are altitudes of $\triangle$ ABC.

38. If, in fig. 600, p. 323, PR and QS are produced to meet at N and if the circles MQS, NRS cut again at X, prove that MXN is a straight line.

[39] The bisectors of $\angle$ ABC, $\angle$ ACB meet at I; CI produced cuts the circle ABC at P. Prove that

(i) $\angle PBI = \frac{1}{2}(\angle ABC + \angle ACB)$; (ii) PB = PI.

**40.** In fig. 608, the centre O of the circle APB lies on the circle AQB, and OPQ is a straight line. Prove that BP bisects ∠ ABQ.

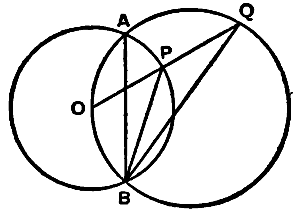


FIG. 608

[41] ACB, ADB are two arcs on the same side of AB and such that the centre of the circle ADB lies on the arc ACB. If a straight line ACD cuts the arcs at C, D, prove that CB = CD.

**42.** In △ ABC, AB = AC. If the circle, centre C, radius CB, cuts AB, AC at D, E, prove that DE is parallel to the line bisecting ∠ ABC.

**43.** O is the centre of the circle ABC; N is the mid-point of AB; NO, produced if necessary, cuts AC at P. Prove that ∠ APN = ∠ OCB. [Produce CO to cut the circle at Q, join QB.]

[44] If O is the centre of the circle in fig. 600, p. 323, and if PS cuts QR at N, prove that ∠ PNR + ∠ PMR = ∠ POR.

**45.** AB, CD are perpendicular chords of a circle, centre O. Prove that ∠ DAB = ∠ OAC.

***46.** The bisectors of the angles ABC, ACB of △ ABC meet at I; the circle BIC cuts AB, AC again at P, Q respectively. Prove that AB = AQ and AC = AP.

***47.** The bisectors of the angles ABC, ACB of △ ABC intersect at I and cut AC, AB at Y, Z respectively; the circles BIZ, CIY meet again at X. Prove that ∠ YXZ + ∠ BIC equals two right angles.

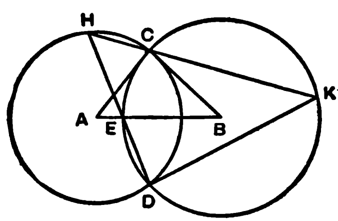


FIG. 609

***48.** In fig. 609, A, B are the centres of the circles DCH, DECK; AEB, DEH, HCK are straight lines. Prove that the triangles HDK, ACB are equiangular.

*49. AKB, CKD are perpendicular chords of a circle ACBD. Prove that the perpendicular from K to AD bisects, when produced, BC.

*50. Two given circles ABP, ABQ intersect at A, B; a variable line through A meets the circles at P, Q, as in fig. 610. Prove that ∠PBQ is of constant size.

*51. In fig. 610, the circles ABP, ABQ are given; PAQ is a variable line through A; PSB, QBT are straight lines; QS meets PT at R. Prove that ∠ PRQ is of constant size.

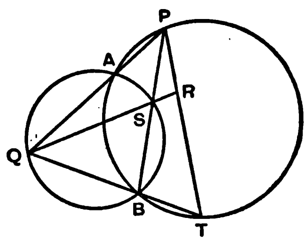



FIG. 610

*52. Two circles PAB, QASB cut at A, B; PAQ, PSB are straight lines, as in fig. 610. If O is the centre of the circle PAB, prove that OP is perpendicular to QS, produced if necessary. [Join AB, AO.]

*53. If in fig. 600, p. 323, QS and PR are produced to meet at N, prove that the bisectors of the angles PMR, QNP cut one another at right angles.

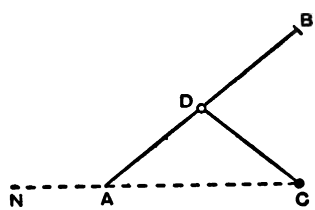



FIG. 611

*54. CD is a rod hinged at a fixed point C and loosely jointed at D to the rod ADB, see fig. 611. If AD = DB = DC and if A moves along the straight line CN, and if D moves in a given plane through AC, prove that B will also move along a straight line.

## Tests for Concyclic Points

Any three points lie either on a straight line or a circle. In general, if any four points are taken in a plane, it is impossible to draw either a straight line or a circle to pass through all of them. We can obtain tests for determining whether four points are concyclic by proving that the converses of Theorems 48, 50 are true.

**(1) If, in fig. 612, $n_1 = n_2$, then the points A, P, Q, B are concyclic.**

If, in fig. 612, a circle is drawn through A, P, B, and if it does not pass through Q it will cut BQ at a point X which lies either on BQ produced or between B and Q or on QB produced, or it will not meet BQ at any other point except B, see fig. 613 (i), (ii), (iii), (iv). It can be *proved* that if $n_1 = n_2$ each of these four alternatives is impossible and hence the circle through A, P, B must also pass through Q. But if we proceed in this way, each of these four cases must be considered.

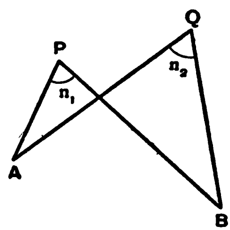


FIG. 612

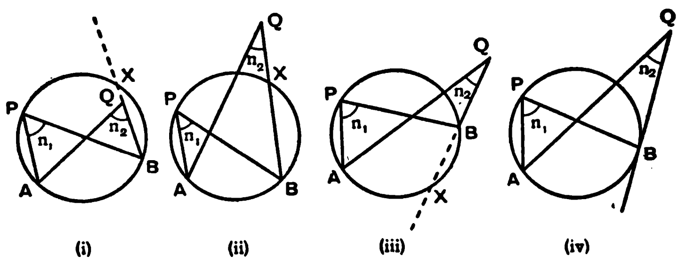


FIG. 613

The object of the argument used in the proof of Theorem 52 is to show that a method can be used in which only two alternative cases need be considered.

**(2) If, in fig. 614, $\angle a + \angle c = 2$ rt. $\angle$s, then the points A, B, C, D are concyclic.**

If, in fig. 614, a circle is drawn through B, C, D, and if it does not pass through A it will cut DA at a point X which lies either on DA produced or between D and A or on AD produced, or it will not meet DA at any other point except D.

Draw four figures to show these alternatives. It can be proved that, if $\angle A + \angle C = 2$ rt. $\angle$s, each of these four alternatives is impossible, and hence the circle through B, C, D must also pass through A. But if we proceed in this way, each of these four cases must be considered. The object of the argument used in the proof of Theorem 53 is to show that a method can be used in which only two alternative cases need be considered.

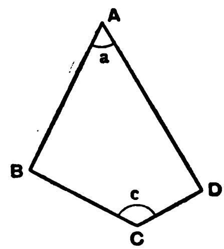


FIG. 614

In connection with this discussion of the various possible forms a construction may take, the reader may examine the following fallacy:

Given *any* triangle ABC, prove that AB = AC.

In fig. 615, the bisector of $\angle$BAC cuts the perpendicular bisector of BC at K; KX, KY are perpendicular to AB, AC; KB, KC are joined.

(i) From $\Delta$s AKX, AKY,
prove that AX = AY
and KX = KY.

(ii) From $\Delta$s BNK, CNK,
prove that KB = KC.

(iii) From $\Delta$s KXB, KYC,
prove that XB = YC.

$\therefore$ AB = AX + XB = AY + YC = AC.

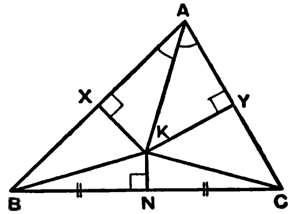


FIG. 615

**What is wrong with this proof?**

## THEOREM 51

**The circle described on the hypotenuse of a right-angled triangle as diameter passes through the opposite vertex.**

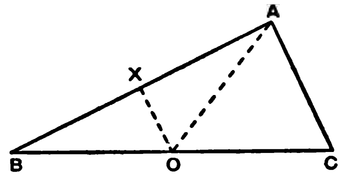



FIG. 616

**Given** a triangle ABC in which ∠BAC = 1 rt. ∠.

**To prove that** the circle on BC as diameter passes through A.

**Construction.** Bisect BC at O and bisect BA at X.

Join OA, OX.

**Proof.** Since BO = OC and BX = XA,

OX is parallel to CA *mid-point theorem.*

But ∠CAB = 1 rt. ∠ *given,*

∴ ∠OXB = 1 rt. ∠ *corr. ∠s,* OX ∥ CA,

∴ OX is the perpendicular bisector of BA,

∴ OB = OA.

But OB = OC *constr.,*

∴ OA = OB = OC,

∴ the circle on BC as diameter passes through A.

**Corollary. The line joining the mid-point of the hypotenuse of a right-angled triangle to the opposite vertex is equal to half the hypotenuse.**

Since OA = OB = OC, OA = $\frac{1}{2}$BC.

NOTE. This theorem is a special case of Theorem 52, making use of Theorem 49.

## THEOREM 52

**If the straight line joining two points subtends equal angles at two other points on the same side of it, then the four points lie on a circle.**

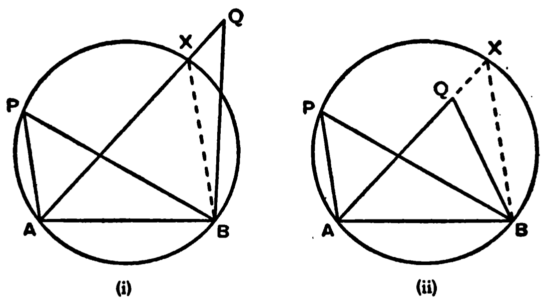



FIG. 617

**Given** two points P, Q on the same side of a straight line AB such that $\angle APB = \angle AQB$.

**To prove that** A, P, Q, B lie on a circle.

**Construction and Proof.** Since $\triangle$s PAB, QAB are on the same side of AB, either the angles PAB, QAB are unequal or the angles PBA, QBA are unequal, because otherwise P would coincide with Q.
Suppose $\angle PAB > \angle QAB$,

then AQ lies in the angle PAB.

Draw the circle through P, A, B and suppose, if possible, it does not pass through Q.
Since AQ lies in the angle PAB, the circle must cut AQ or AQ produced, at X, say.
Join BX.

Then $\angle APB = \angle AXB$ *$\angle$s in same segment,*
but $\angle APB = \angle AQB$ *given,*
$\therefore \angle AXB = \angle AQB$.

But one of these is the exterior angle and the other is the interior opposite angle of $\triangle$ BQX, therefore they cannot be equal.

Therefore the original supposition is false.

$\therefore$ the circle through P, A, B must pass through Q.

**Corollary.** **Given the base AB of a triangle ABP in magnitude and position, and given the size of $\angle$APB, then the locus of P is two equal arcs of equal circles on opposite sides of AB.**

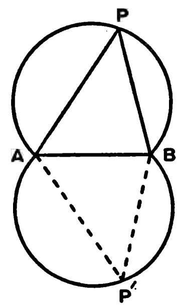


FIG. 618

Hence it follows that, if the length of one side of a triangle and the size of the angle *opposite* to that side are given, the **circumradius** of the triangle is fixed and can be found either by measurement or by using trigonometry. Let O be the circumcentre of $\triangle$ABC and R its circumradius. Draw the diameter BOK of the circumcircle; join AK. In fig. 619 (i), explain why $c = \text{BK} \sin \text{C}$; deduce that $\text{R} = \dfrac{c}{2 \sin \text{C}}$.

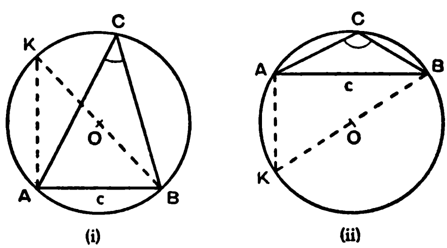


FIG. 619

In fig. 619 (ii), explain why $c = \text{BK} \sin (180° - \text{C})$ and (see p. 246) deduce that $\text{R} = \dfrac{c}{2 \sin \text{C}}$. Write down two similar expressions for R, compare p. 251, No. 28.

## THEOREM 53

**If a pair of opposite angles of a quadrilateral are supplementary, its vertices are concyclic.**

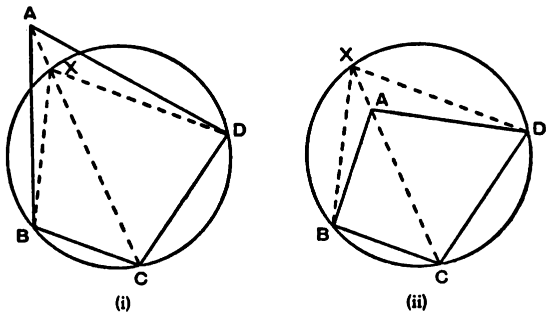



FIG. 620

**Given** a quadrilateral ABCD in which

$$\angle ABC + \angle ADC = 2 \text{ rt. } \angle s.$$

**To prove that** A, B, C, D lie on a circle.

**Construction and Proof.** Draw the circle through B, C, D and suppose, if possible, that it does not pass through A. Since CA lies in the angle BCD, the circle BCD must cut either CA or CA produced, at X, say.
Join XB, XD.

$\angle XBC + \angle XDC = 2 \text{ rt. } \angle s$ *opp. ∠s, cyclic quad.*,

$\angle ABC + \angle ADC = 2 \text{ rt. } \angle s$ *given*,

$\therefore \angle XBC + \angle XDC = \angle ABC + \angle ADC.$

But one side of this equation is part of the other side, therefore the two sides cannot be equal.
Therefore the original supposition is false.

$\therefore$ **the circle through B, C, D must pass through A.**

# NUMERICAL EXAMPLES

## EXERCISE 60

**1. In fig. 621, find whether the points A, B, C, D are concyclic** if (i) $m = 130°$, (ii) $m = 140°$.

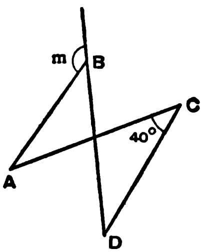



FIG. 621

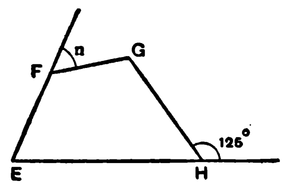



FIG. 622

2. In fig. 622, find whether the points E, F, G, H are concyclic if (i) $n = 55°$, (ii) $n = 45°$.

3. Draw a quadrilateral PQRS and its diagonals PR, QS. If $\angle PQR = 70°$, $\angle PRQ = 35°$, $\angle QSR = 75°$, prove that P, Q, R, S are concyclic and find $\angle PSQ$.

[4] The diagonals AC, BD of the quadrilateral ABCD intersect at K. If $\angle BAC = 50°$, $\angle CAD = 45°$, $\angle ACD = 55°$, $\angle BKC = 105°$, prove that A, B, C, D are concyclic and find $\angle CBD$.

[5] ABCD is a quadrilateral in which $\angle ABD = 30°$, $\angle ADB = 40°$, $\angle BCD = 70°$. Find $\angle ACB$.

6. Draw a quadrilateral ABCD and its diagonals AC, BD. If $\angle DAC = 65°$, $\angle CAB = 50°$, $\angle CBD = 65°$, state the sizes of any other angles in the figure that can be calculated.

7. ABCD is a quadrilateral in which $AB = AD$ and $DB = DC$. If $\angle DBA = x°$ and $\angle DBC = 2x°$, prove that A, B, C, D are concyclic.

[8] BE, CF are altitudes of $\triangle ABC$. If $\angle AEF = 65°$, find $\angle BCF$.

9. P, Q, R are points on the sides BC, CA, AB of $\triangle$ ABC such that $\angle$ RPB $=30°$, $\angle$ QPC $=20°$, $\angle$ PRQ $=10°$. If $\angle$ ABC $=55°$ and $\angle$ ACB $=75°$, prove that PQAR and BRQC are cyclic quadrilaterals.

10. AD, BE are altitudes of $\triangle$ ABC. If $\angle$ ADE $=30°$ and $\angle$ BED $=20°$, find the angles of $\triangle$ ABC.

[11] In $\triangle$ ABC, AB = AC and $\angle$ BAC $=36°$. If the line bisecting $\angle$ ABC meets the line through C parallel to BA at D, prove that A, B, C, D are concyclic.

12. A, B are fixed points such that AB $=4$ cm.; P is a variable point such that $\angle$ APB $=70°$. Construct the complete locus of P. [See p. 314, No. 2 (iv).]

[13] In $\triangle$ ABC, BC $=2$ in. and $\angle$ BAC $=30°$. Find the radius of the circle ABC.

14. Construct $\triangle$ ABC given that $\angle$ ACB $=50°$, AB $=5$ cm. and that the distance of C from AB is 4 cm.

## EXERCISE 61

Nos. 1–5 refer to fig. 623 in which BE, CF are altitudes of $\triangle$ ABC and intersect at H.

1. Prove that (i) B, F, E, C are concyclic; (ii) $\angle$ AEF $=\angle$ ABC.

[2] Prove that (i) A, E, H, F are concyclic; (ii) $\angle$ AHE $=\angle$ ACB.

3. Prove that $\angle$ HAE $=\angle$ EBC and deduce that AH produced cuts BC at right angles.

4. If X is the mid-point of BC, prove that XE $=$ XF.

[5] If X is the mid-point of BC, prove that $\angle$ FXE $=180° - 2\angle$ BAC.

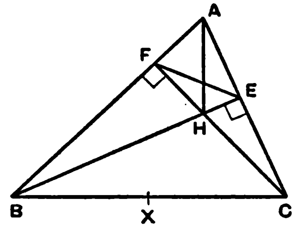


FIG. 623

6. ABCD is a parallelogram; any circle through A and D cuts AB, DC at P, Q. Prove that B, C, Q, P are concyclic.

[7] ABCD is a parallelogram in which $\angle$ ABC $=60°$. Prove that the centre of the circle ABD lies on the circle CBD.

8. In fig. 624, P, Q, R are points on BC, CA, AB. Prove that AQKR is a cyclic quadrilateral. [Join KP.]

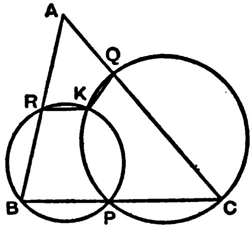



FIG. 624

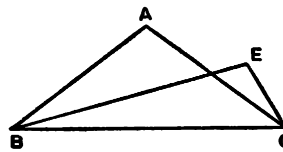



FIG. 625

9. In fig. 625, $AB = AC$ and $\angle ABE = \angle ACE$. Prove that $\angle AEC = 90° + \frac{1}{2}\angle BAC$.

[10] If in fig. 628, p. 338, AC is a diameter of the circle and if $AD = DQ$, prove that (i) $CA = CQ$; (ii) $DB = DQ$.

[11] In $\triangle ABC$, $AB = AC$ and $BC > AB$; P is a point on CA produced and Q is a point on BC such that $\angle BQP = 2\angle QPC$. Prove that A, P, B, Q are concyclic.

12. In fig. 626, O is the centre of the circle and OLM is perpendicular to AOB. Prove that (i) A, O, M, P are concyclic; (ii) $\angle OPA = \angle OMB$.

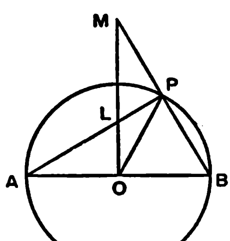



FIG. 626

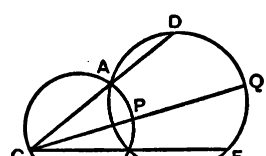



FIG. 627

13. In fig. 627, CAD, CPQ, CBE are straight lines. If DE cuts CQ at K, prove that A, P, K, D are concyclic. [Join AP, AB.]

[14] In $\triangle ABC$, $AB > AC$ and $P$ is a point on $AB$ such that $AP = AC$. The bisector of $\angle BAC$ cuts $BC$ at $Q$ and cuts the circle $ABC$ at $R$. Prove that $B, P, Q, R$ are concyclic.

[15] $A, B, C$ are any three points on a circle. The internal and external bisector of $\angle BAC$ cut the circle again at $H, K$. Prove that $HK$ is a diameter of the circle.

[16] Prove that the quadrilateral formed by the external bisectors of the angles of any quadrilateral is cyclic.

17. In fig. 628, $PBA$, $PCD$, $QDA$, $QCB$ are straight lines. If $\angle APD = \angle BQA$, prove that $AC$ is a diameter.

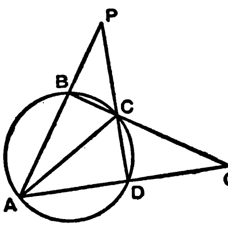



FIG. 628

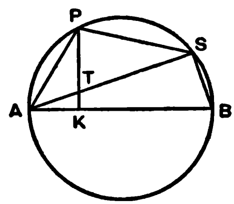



FIG. 629

18. In fig. 629, $AB$ is a diameter of the circle. If $\angle APK = \angle ASP$, prove that (i) $\angle PKB = 90°$; (ii) $STKB$ is a cyclic quadrilateral. [Join $PB$.]

[19] The base $AB$ and the angle $ACB$ of a triangle $ABC$ are given. If $\angle ACB$ is acute, prove that $AC$ is greatest when $\angle ABC = 90°$.

[20] The circle $BCGF$ lies inside the circle $ADHE$; $OABCD$ and $OEFGH$ are two straight lines cutting the circles. If $A, B, F, E$ are concyclic, prove that $C, D, H, G$ are concyclic.

21. $PQ$, $PR$ are any two chords of a circle, centre $O$. If the diameter perpendicular to $PQ$ cuts $PR$ at $K$, prove that $Q, O, K, R$ are concyclic.

[22] $ABCD$ is a cyclic quadrilateral; $AP$, $DQ$ are the perpendiculars from $A$, $D$ to $CD$, $AB$ respectively. Prove that $PQ$ is parallel to $CB$

23. In fig. 630, the side BA of the equilateral triangle ABC is produced to Y, and AX is parallel to BC. If $\angle CYX = 60°$, prove that $\triangle CXY$ is equilateral.

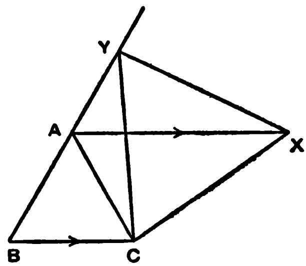



FIG. 630

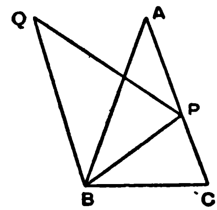



FIG. 631

24. In fig. 631, $AB = AC$, $QB = QP$ and $\angle BAC = \angle BQP$. Prove that QA is parallel to BC.

[25] Two circles APRB, ASQB intersect at A, B; PAQ and RAS are straight lines; RP and QS are produced to meet at O. Prove that O, P, B, Q are concyclic.

*26. AOB, COD are two perpendicular diameters of a circle. Two chords CP, CQ cut AB at H, K. Prove that H, K, Q, P are concyclic. [Join CA, CB, AQ, PQ.]

*27. If any five circular arcs are drawn intersecting as in fig. 632, prove that a sixth circle can be drawn to pass through P, Q, R, S.

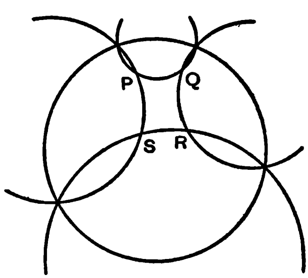



FIG. 632

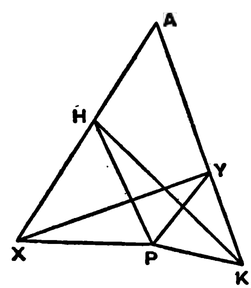



FIG. 633

*28. In fig. 633, $\triangle PXY \equiv \triangle PHK$ and the lines XH, KY meet when produced at A. Prove that the circle XPY passes through A.

*29. X, Y are the centres of the circles ABP, ABQ; PAQ is a straight line; PX and QY are produced to meet at R. Prove that X, Y, B, R are concyclic. [Join AX, AY, BX, BY.]

*30. In fig. 624, p. 337, if K is a given point inside a given triangle ABC, prove that the angles of $\triangle$ PQR are of constant size.

*31. ABCD is a parallelogram; O is a point inside ABCD such that $\angle$AOB + $\angle$COD = 2 rt. $\angle$s. Prove that $\angle$OBC = $\angle$ODC. [Draw AQ, BQ∥DO, CO; join OQ.]

*32. OX, OY are given perpendicular lines. ABP is a triangle such that AB = 5 in., BP = 3 in., PA = 4 in. If the triangle moves in the plane XOY so that A slides along OX and B slides along OY, with O and P on opposite sides of AB, prove that the locus of P is part of a straight line. [Join OP.]

## Equal Arcs of the same or Equal Circles

Fig. 634 represents two minor arcs AB, PQ of a circle ABPQ, centre O, such that $\angle$AOB = $\angle$POQ.

If we draw the diameter XOY which bisects $\angle$BOP, it also bisects $\angle$AOQ. But the circle is symmetrical about the diameter XOY, see p. 305; therefore if we fold the figure about XOY, B can be made to coincide with P, and A with Q, and the minor arc AB will coincide with the minor arc QP. Thus

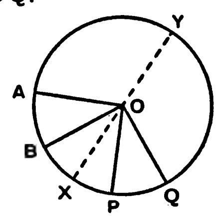


Fig. 634

**If O is the centre of the circle ABPQ and if $\angle$AOB = $\angle$POQ, then**

**minor arc AB = minor arc PQ.**

Suppose an arc AB of a circle, centre O, is given. What measurements must be taken in order to copy the arc AB?

If the radius of the circle is measured, a copy of the circumference of the circle can be made.

Since arcs of a circle which subtend equal angles at the centre are of equal length, the length of the arc AB is fixed if the size of $\angle$AOB is known. Therefore, if $\angle$AOB is measured, a copy of the arc can be made.

Consequently if two circular arcs are drawn which agree with each other as regards

(i) the length of the radius,

and (ii) the angle subtended at the centre of the circle,

then they will agree completely in size and shape, in other words they will be **congruent.**

Thus, see fig. 635,

**If H, K are the centres of two equal circles ABP, CDQ, and if $\angle$AHB = $\angle$CKD, then minor arc AB = minor arc CD;**

and conversely,

**If H, K are the centres of two equal circles ABP, CDQ, and if arc AB = arc CD, then $\angle$AHB = $\angle$CKD.**

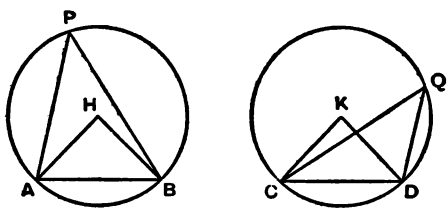



FIG. 635

## Examples for Oral Discussion

**1.** Fig. 635 represents two equal circles, centres H, K.

(i) If arc AB = arc CD, prove that $\angle$APB = $\angle$CQD.
(ii) If $\angle$APB = $\angle$CQD, prove that minor arc AB = minor arc CD.

**2.** Fig. 635 represents two equal circles, centres H, K.

(i) If arc AB = arc CD, prove that chord AB = chord CD.
(ii) If chord AB = chord CD, prove that minor arc AB = minor arc CD.

**3.** If two arcs of two circles are of equal length, must the arcs be congruent?

## Calculation of Length of Arc

If the circumference of a circle is divided into 360 equal parts, each arc subtends an angle of 1° at the centre. Hence the length of the arc which subtends an angle of $x°$ at the centre is $\frac{x}{360}$ of the circumference.

If the radius of a circle is $r$ in., the length of the circumference is $2\pi r$ in. where $\pi \simeq 3{\cdot}1416$.

Hence **the length of an arc** of a circle of radius $r$ in., which subtends an angle $x°$ at the centre is $\mathbf{\frac{x}{360} \times 2\pi r}$ **in.**

## Calculation of Area of Sector

If the area of a circle is divided into 360 sectors of equal area, the angle of each sector, *i.e.* the angle between the radii bounding it, is 1°. Hence the area of a sector of angle $x°$ is $\frac{x}{360}$ of the area of the circle.

If the radius of a circle is $r$ in., the area of the circle is $\pi r^2$ sq. in. where $\pi \simeq 3{\cdot}1416$.

Hence **the area of a sector**, angle $x°$, of a circle of radius $r$ in. is $\mathbf{\frac{x}{360} \times \pi r^2}$ sq. in.

$\therefore$ **area of sector $= \frac{1}{2}$ radius $\times$ length of arc.**

## Area of the Curved Surface of a Circular Cone

Let the slant height of the cone be $l$ in., and the radius of the base of the cone be $r$ in.

If we cut down a slant edge of the cone and fold out flat the curved surface, we obtain a sector of a circle, radius $l$ in., length of arc $2\pi r$ in.

But the area of this sector is $\frac{1}{2}l \times 2\pi r$ sq. in.,

$\therefore$ **area of curved surface of cone $= \pi rl$ sq. in.**

The results which have been discussed are expressed by the following theorems which are proved in the Appendix, pp. 548–9.

### THEOREM 54

**(i) In equal circles (or in the same circle), equal angles at the centres (or centre) stand on equal arcs.**

**(ii) In equal circles (or in the same circle), equal angles at the circumferences (or circumference) stand on equal arcs.**

### THEOREM 55

**(i) In equal circles (or in the same circle), equal arcs subtend equal angles at the centres (or centre).**

**(ii) In equal circles (or in the same circle), equal arcs subtend equal angles at the circumferences (or circumference).**

## NUMERICAL EXAMPLES

### EXERCISE 62 (Oral)

Nos. 1–8 refer to fig. 636, in which arc AB $=\frac{1}{10}$ circumference, arc AC $=\frac{1}{6}$ circumference, arc APD $=\frac{5}{12}$ circumference. Find the following angles:—

1. $\angle$AQB.
2. $\angle$BQC.
3. $\angle$APD.
4. $\angle$CPD.

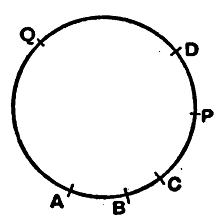


FIG. 636

Express as fractions of the circumference:

5. Minor arc BC.
6. Major arc CQD.

7. Find the ratio of minor arc AB to minor arc BC.

8. If minor arc AQ = twice arc QD, find $\angle$QAD.

Nos. 9–16 refer to fig. 637.

**9.** What fraction of the circumference is the minor arc BC?

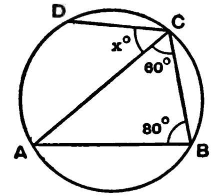


FIG. 637

**10.** Prove arc ADC = twice minor arc BC.

**11.** Find the ratio of minor arc AB to minor arc BC.

**12.** Find $x$ if arc AD = twice arc DC.

**13.** Find $x$ if arc CD = half arc CB.

**14.** Find $x$ if arc AB + arc CD = arc AD + arc BC.

**15.** If the radius of the circle is 5 cm., find correct to 3 figures the lengths of the minor arcs AB and BC.

**16.** If the length of the arc ADC is 8 cm., find correct to 2 figures the radius of the circle. [Take $\pi = \frac{22}{7}$.]

# NUMERICAL EXAMPLES

## EXERCISE 63

[*In this exercise, unless otherwise stated, take* $\pi = 3{\cdot}142$ *and give those answers which depend on the value of* $\pi$ *to three figures.*]

**1.** The value of $\pi$ is found experimentally by wrapping a piece of thread 5 times round a cylinder of diameter 5 inches. When unwrapped the thread measures 78·65 in. What value does this give for $\pi$?

Find the lengths of the circumferences of the circles Nos. 2–4:

**2.** Radius, 4 in. **3.** Diameter, 7 cm. [4] Radius, 100 yd.

Find the radii of the circles, Nos. 5–8, to 2 figures. Take $\pi = \frac{22}{7}$.

**5.** Circumference, 11 in. [6] Circumference, 8·8 cm.

**7.** Circumference, 440 yd. [8] Circumference, 6 ft.

[9] The diameter of a semi-circular protractor is 3·5 in. Find its perimeter, correct to $\frac{1}{10}$ in

[10] Find the length of an arc of a circle of radius 4 cm., which subtends 50° at the centre.

11. ABC is an equilateral triangle inscribed in a circle of radius 4 cm. Find the length of the major arc ABC.

[12] If the length of an arc of a circle of radius 5 cm. is 4 cm., find to the nearest degree the angle which the arc subtends at the centre. [Take $\pi = \frac{22}{7}$.]

13. If a circular arc, 6 cm. long, subtends 80° at the centre, find the radius of the circle correct to 2 figures. [Take $\pi = \frac{22}{7}$.]

14. A circle of radius 2 in. is drawn on squared paper. By counting squares, a boy estimates the area of the circle to be 12·57 sq. in. What value does this give for $\pi$?

Find the areas of the circles, Nos. 15–17:

15. Radius, 10 in. [16] Diameter, 8 cm. [17] Radius, 7 ft.

Find the radii of the circles, Nos. 18, 19, correct to 2 figures. Take $\pi = \frac{22}{7}$.

18. Area, 616 sq. in. [19] Area, 38·5 sq. cm.

[20] Find the area of the ring between two concentric circles of radii 3 in. and 4 in.

21. The angle of a sector of a circle, radius 2·5 cm., is 108°. Find the area of the sector.

22. A square ABCD is inscribed in a circle of radius 4 in. Find the area of the minor segment cut off by AB.

23. AB is a chord of a circle AKB, radius 4 in. If $\angle$ AKB = 30°, find the area of the segment AKB.

---

24. ABCD is a square and AEF is an equilateral triangle inscribed in the circle ABECFD. Find the angles of $\triangle$ ECD.

[25] ABCDE is a regular pentagon inscribed in a circle. Find the angles of $\triangle$ ABD.

[26] AB is a side of a regular hexagon and AC of a regular octagon, inscribed in the same circle. Find the angles of $\triangle$ ABC. [Two sets of answers.]

27. Find the angles of the triangle formed by joining the points II, VI, IX on the face of a clock.

28. A, B are points on the circle ABCD such that the minor arc AB is half the major arc AB; $\angle$ DAB = 74°; arc BC = arc CD. Find $\angle$ ABD and $\angle$ BDC.

[29] A, B, C are 3 points on a circle such that $\angle ABC = 38°$, $\angle ACB = 68°$; P, Q are the mid-points of the minor arcs AC, AB respectively. Find $\angle BCP$ and $\angle CPQ$.

30. ABCD is a square and APQ is an equilateral triangle inscribed in the circle ABPCQD. Prove that arc $BP = \frac{1}{2}$ arc PC.

[31] ABC is a triangle inscribed in a circle; T is a point on BC produced. If $\angle BAC = 120°$, $\angle CAT = 15°$, $\angle ATB = 30°$, find the ratio of the arc AB to the arc AC.

32. On a clock-face, prove that the line joining the points IV, VII is perpendicular to the line joining the points V, XII.

33. If, in fig. 638, arc BD is four times the arc AC, find $\angle ADC$.

34. With the data of fig. 638, find what fraction arc AD + arc BC is of the circumference. [Join AC.]

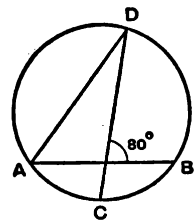


FIG. 638

*35. Draw a circle of radius 5 cm. and place in it a chord AB of length 4 cm. Find the area of the major segment cut off by AB, making any necessary measurements.

*36. A piece of wire is in the form of an arc of a circle of radius 6 cm., subtending 150° at the centre. It is bent into a complete circle. Find the radius of the circle.

*37. A piece of wire 1 yd. long is bent into the form of a semi-circular arc and its diameter. Find the radius. [Take $\pi = \frac{22}{7}$.]

*38. Find the radius of a circle whose area is equal to the sum of the areas of two circles of radii, 3 in. and 4 in.

*39. ABCD is a quadrilateral inscribed in a circle. If $\angle ADB = 25°$ and $\angle DBC = 65°$, prove that

arc AB + arc CD = arc BC + arc AD.

*40. ABC is an equilateral triangle and PQRS is a square inscribed in the circle APQBRCS. If AB is parallel to PQ, prove that A, P, Q, B, R, C, S are some of the vertices of a regular 24-sided polygon.

*41. ABCD is a quadrilateral inscribed in a circle; AC cuts BD at K; DA, CB when produced meet at E; AB, DC when produced meet at F. If $\angle AEB = 55°$, $\angle BFC = 35°$, $\angle DKC = 85°$, prove that arc BC is twice arc AB.

## THEOREM 56

**In equal circles (or in the same circle) equal chords cut off equal arcs.**

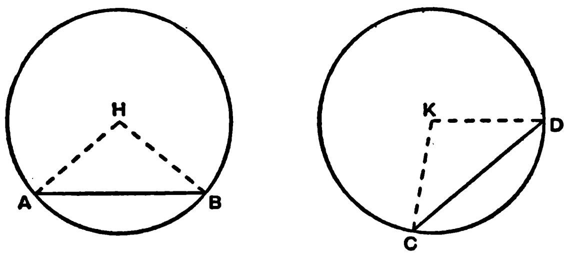



Fig. 639

**Given** two equal circles, centres H, K, and two equal chords AB, CD of these circles.

**To prove that** minor arc AB = minor arc CD,
and major arc AB = major arc CD.

**Construction.** Join HA, HB, KC, KD.

**Proof.** In △s HAB, KCD,

HA = KC *radii of equal circles, given,*
HB = KD *radii of equal circles, given,*
AB = CD *given,*

$\therefore$ △s $\begin{matrix} HAB \\ KCD \end{matrix}$ are congruent SSS.

$\therefore$ ∠AHB = ∠CKD.

But, in equal circles, equal angles at the centres stand on equal arcs,

$\therefore$ minor arc AB = minor arc CD.

But the circumference of the two circles are equal,

$\therefore$ major arc AB = major arc CD.

## THEOREM 57

**In equal circles (or in the same circle) the chords of equal arcs are equal.**

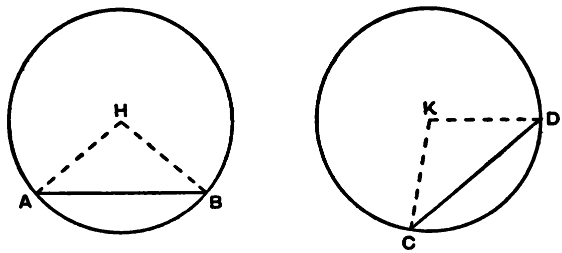



FIG. 640

**Given** two equal circles, centres H, K, and two equal arcs AB, CD of these circles.

**To prove that** chord AB = chord CD.

**Construction.** Join HA, HB, KC, KD.

**Proof.** In equal circles the angles at the centres which stand on equal arcs are equal,

$\therefore \angle AHB = \angle CKD,$

$\therefore$ in $\triangle$s HAB, KCD,

| | |
|---|---|
| $HA = KC$ | *radii of equal circles, given,* |
| $HB = KD$ | *radii of equal circles, given,* |
| $\angle AHB = \angle CKD$ | *proved,* |

$\therefore \triangle s \begin{matrix} HAB \\ KCD \end{matrix}$ are congruent SAS.

$\therefore AB = CD.$

Theorems 54–57 give useful tests for the equality of arcs or of chords or of angles.

Two arcs of a circle may be proved equal by showing they subtend equal angles at the circumference (or at the centre).

Two chords of a circle may be proved equal by showing they subtend equal *or* supplementary angles at the circumference.

Two angles at the circumference of a circle may be proved equal by showing that they stand on equal arcs.

*These tests also hold for* equal *circles.*

### Examples for Oral Discussion

**1.** What deductions can you make in fig. 641 if it is given that AX bisects ∠BAC?

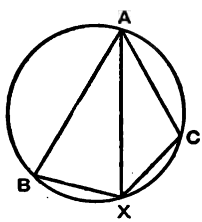


FIG. 641

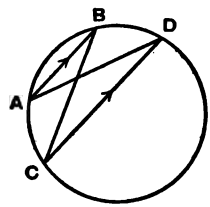


FIG. 642

**2.** In fig. 642, AB, CD are parallel chords of the circle. Prove that

(i) Arc AC = arc BD; (ii) chord AD = chord BC.

(i) What angles must you prove equal?

(ii) What arcs must you prove equal?

**3.** In fig. 643 the circles are equal and PAQ is a straight line. Prove that BP = BQ.

Join AB. Complete the statement: In equal circles, the angles which stand on equal chords are either equal or . . .

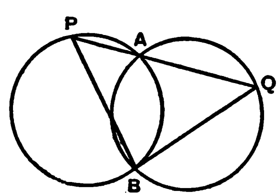


FIG. 643

## THEOREM 57

**In equal circles (or in the same circle) the chords of equal arcs are equal.**

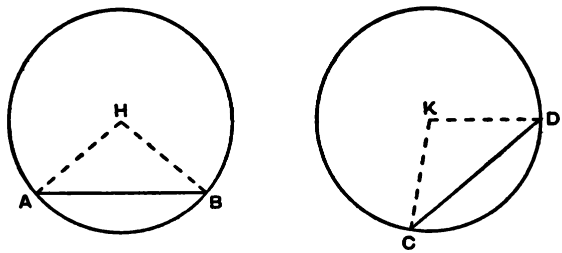



FIG. 640

**Given** two equal circles, centres H, K, and two equal arcs AB, CD of these circles.

**To prove that** chord AB = chord CD.

**Construction.** Join HA, HB, KC, KD.

**Proof.** In equal circles the angles at the centres which stand on equal arcs are equal,

$\therefore$ $\angle AHB = \angle CKD$,

$\therefore$ in $\triangle$s HAB, KCD,

| | |
|---|---|
| HA = KC | *radii of equal circles, given,* |
| HB = KD | *radii of equal circles, given,* |
| $\angle AHB = \angle CKD$ | *proved,* |

$\therefore$ $\triangle$s $\begin{matrix} HAB \\ KCD \end{matrix}$ are congruent SAS.

$\therefore$ AB = CD.

Theorems 54–57 give useful tests for the equality of arcs or of chords or of angles.

Two arcs of a circle may be proved equal by showing they subtend equal angles at the circumference (or at the centre).

Two chords of a circle may be proved equal by showing they subtend equal *or* supplementary angles at the circumference.

Two angles at the circumference of a circle may be proved equal by showing that they stand on equal arcs.

*These tests also hold for* equal *circles.*

### Examples for Oral Discussion

1. What deductions can you make in fig. 641 if it is given that AX bisects $\angle$BAC?

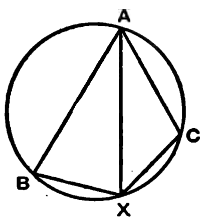


Fig. 641

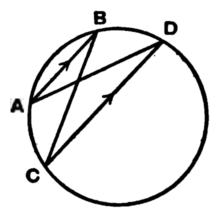


Fig. 642

2. In fig. 642, AB, CD are parallel chords of the circle. Prove that

(i) Arc AC = arc BD; (ii) chord AD = chord BC.

(i) What angles must you prove equal?
(ii) What arcs must you prove equal?

3. In fig. 643 the circles are equal and PAQ is a straight line. Prove that BP = BQ.

Join AB. Complete the statement: In equal circles, the angles which stand on equal chords are either equal or . . .

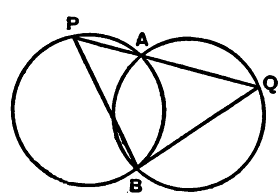


Fig. 643

## EXERCISE 64

1. AB, CD are equal chords of the circle ABCD. Prove that (i) AC = BD; (ii) BC is parallel to AD.

[2] ABCD is a cyclic quadrilateral. If AB = CD, prove that ∠ ABC = ∠ BCD. [On what arcs do these angles stand?]

3. ABCDEF is a hexagon inscribed in a circle.
If ∠ ABC = ∠ DEF, prove that AF is parallel to CD.
[Join AD. What arcs are equal?]

4. A, B, C, D, E are five consecutive vertices of a regular polygon, with more than 5 sides, inscribed in a circle CDEPAB, centre K. Prove that ∠ CPE = ∠ AKB.

[5] The chords AB, DC of a circle ABCD meet when produced at E. If AB = CD, prove that EA = ED.

6. ABCD is a rectangle inscribed in a circle; DP is a chord equal to DC. Prove that (i) arc DP = arc AB; (ii) PB = AD.

7. In fig. 644, X, Y are the mid-points of the arcs AB, AC. Prove that AP = AQ. [Join AX, AY.]

[8] A circle AOBP passes through the centre O of a circle ABQ. Prove that OP bisects ∠ APB.

[9] PQ, RS are parallel chords of the circle PRSQ, centre O. If PS cuts QR at K, prove that ∠ PKR = ∠ POR.

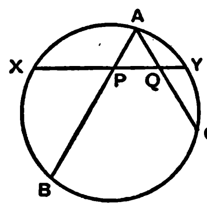



FIG. 644

10. In fig. 645, the chords BX, CY are perpendicular to AC, AB respectively. Prove that arc AX = arc AY.

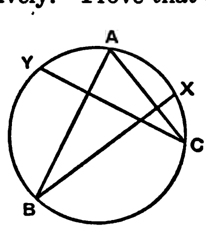



FIG. 645

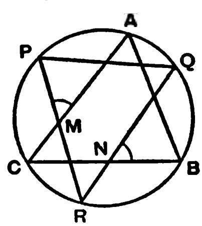



FIG. 646

[11] If, in fig. 646, ∠ PMA = ∠ QNB, prove that PQ = AB. [PQ is not parallel to BC.]

**12.** APQB is an arc of a circle. If the bisectors of $\angle$PAQ, $\angle$PBQ meet at R, prove that R lies on the circle.

[13] ABCD is a quadrilateral inscribed in a circle; CD is produced to F; the bisector of $\angle$ABC cuts the circle at E. Prove that DE bisects $\angle$ADF.

**14.** A hexagon is inscribed in a circle. If two pairs of opposite sides are parallel, prove that the third pair are also parallel. [Let the lengths of the minor arcs cut off by the sides be $a$, $b$, $c$, $d$, $e$, $f$. Use Example 2, p. 349.]

**15.** The side AD of the cyclic quadrilateral ABCD is produced to E so that DE = AB. If AC bisects $\angle$BAD, prove that CE = CA.

[16] The bisectors of $\angle$ABC, $\angle$ACB meet at I, and the circle BIC cuts AB, AC, produced if necessary, at P, Q. Prove that PI = IC and QI = IB.

**17.** In fig. 647, the circles are equal and BXY is a straight line. Prove that AX = AY.

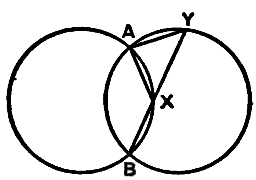


FIG. 647

**18.** P is any point on a chord BC of a circle, centre O. Prove that the circles OPB, OPC are equal.

[19] In $\triangle$ABC, AB = AC; D is any point on BC produced. Prove that the circles ADB, ADC are equal.

**20.** ABCD is a cyclic quadrilateral. If $\angle$ABC = 2$\angle$BAC and if AB is parallel to DC, prove that AD = DC = CB.

[21] A variable triangle PQR is inscribed in a given circle. If the angle QPR is of given size, what else can you say about $\triangle$PQR?

[22] CD is a quadrant of the circle ACDB; AB is a diameter. If AD cuts BC at P, prove that AC = CP.

*23. ABCD is a cyclic quadrilateral; BC and AD meet when produced at E. If the circle ACE cuts AB, CD, produced if necessary, at P, Q, prove that EP = EQ.

*24. AB, BC are two chords of a circle, AB > BC. The minor arc AB is folded over about the chord AB and cuts AC at D. Prove that BD = BC.

*25. ABC is an equilateral triangle inscribed in a circle; H, K are the mid-points of the minor arcs AB, AC. Prove that HK is trisected by AB, AC. [Join AH, AK, CH, BK.]

*26. ABCD is a quadrilateral inscribed in a circle; X, Y, Z, W are the mid-points of the minor arcs AB, BC, CD, DA respectively. Prove that XZ is perpendicular to YW. [Join XY, YZ.]

*27. ABC is a triangle inscribed in the circle BPACQ, centre O; PQ is the diameter perpendicular to BC. Prove that $\angle ACB - \angle ABC = \angle AOP$.

*28. ABC is an equilateral triangle inscribed in a circle; D, E are points on the minor arcs AB, BC such that AD = BE. Prove that AD + DB = AE. [Draw DK parallel to BE to meet AE at K.]

*29. In fig. 648, the three circles are equal, AD = BC, and XAB, DCY are straight lines. Prove that XBYD is a parallelogram.

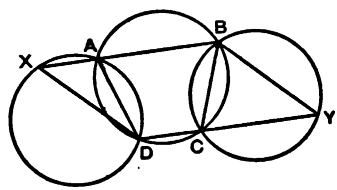



FIG. 648

*30. Two fixed circles cut at A, B; P is a variable point on one circle; PA, PB when produced cut the other circle at Q, R. Prove that QR is of constant length.

## Secants and Tangents

**Definitions.** If a straight line cuts a circle at two distinct points, it is called a **secant**.

If a straight line has one point, and only one point, in common with a circle, however far either way it is produced, the straight line is called a **tangent** to the circle, and the common point is called the **point of contact.**

The words "touching" and "meeting" must not be confused. If a line meets a circle, it may when produced meet it at a second distinct point, and if it does so, the line is a secant. It may, however, have only one point in common with the circle, however far either way it is produced, and if this is so, the line is a tangent.

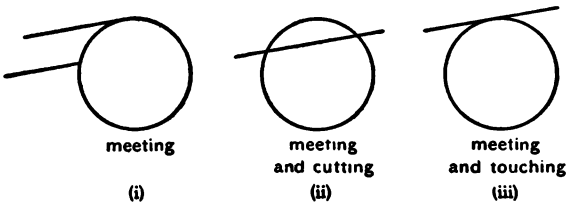



FIG. 649

A discussion of the properties of tangents by the use of limits is given in the Appendix, pp. 550–552.

### Examples for Oral Discussion

**1. A is any point on a circle, centre O, and BAC is drawn at right angles to OA through A. Prove that BAC is a tangent to the circle.**

If P is any point, other than A, on the line BAC, prove that $OP > OA$, and complete the proof.

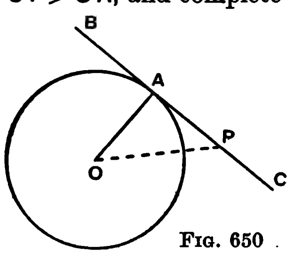



FIG. 650

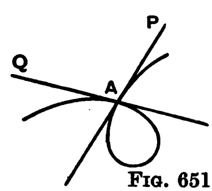



FIG. 651

If we *assume* that there is only one tangent to a circle at any given point on it, it follows that the converse of Example 1 is true. It is, however, easy to draw curves for which this assumption is untrue, see fig. 651. It is therefore desirable to prove that this assumption is true for a circle.

**2. If a straight line BAC touches at A the circular arc AEF, centre O, prove that OA is perpendicular to BAC.**

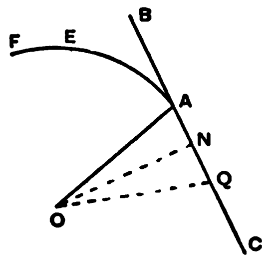



FIG. 652

*If possible*, suppose OA is not perpendicular to BAC and let ON be the perpendicular from O to BC. Produce AN to Q so that AN = NQ. Join OQ and prove that OQ = OA.

Explain why this is contrary to the data.

**3. T is any point outside a given circle, centre O. Construct the tangents from T to the circle.**

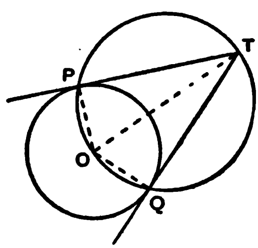



FIG. 653

Let the circle on TO as diameter cut the given circle at P, Q. Then TP, TQ are the required tangents.

(i) Explain why ∠OPT is a right angle.

(ii) Prove that TP and TQ are tangents.

**4. If TP, TQ are the tangents from T to a circle, centre O, prove that (i) TP = TQ; (ii) OT bisects ∠PTQ and ∠POQ.**

Explain why △OPT ≡ △OQT.

**Length of Tangent.** If a line TP is drawn from a point outside a given circle to touch the circle at P, the distance of T from the *point of contact* P is called the *length of the tangent* from T to the circle.

Example 4 proves that the tangents to a circle from any external point are equal. In speaking of the length of the tangent from a point to a circle, it is therefore unnecessary to say which of the two tangents is taken.

5. If, in fig. 653, the radius of the circle is 3 cm. and the distance of T from the centre O is 5 cm., calculate the length of the tangent from T to the circle.

## NUMERICAL EXAMPLES

### EXERCISE 65

Nos. 1–4 refer to fig. 654 in which TAB is the tangent at A to the circle, centre O.

1. If $\angle PAB = 25°$, find $\angle OAP$ and $\angle AOP$.

[2] If $\angle AOP = 70°$, find $\angle PAB$.

3. If $OA = 6$ cm., $OT = 10$ cm., find TA.

[4] If $TA = 12$ cm., $OA = 5$ cm., find OT.

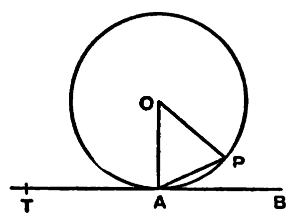


FIG. 654

Nos. 5–7 refer to fig. 653, p. 354, in which TP, TQ are the tangents from T to the circle, centre O.

5. If $\angle PTQ = 70°$, find $\angle POQ$.

[6] If $\angle TOP = 56°$, find $\angle PTQ$.

7. If $\angle PTQ = 36°$, find $\angle TPQ$.

8. ABC is a minor arc of a circle; the tangents at A, C meet at T. If ∠ATC = 54°, find ∠ABC.

Nos. 9–11 refer to fig. 655 in which AC is a diameter of the circle and TAB is a tangent.

9. If ∠QAB = 38°, find ∠QAC and ∠QCA.

[10] If ∠APQ = 42°, find ∠QAB.

11. If ∠QAT = 155°, find the angle in the minor segment AQ.

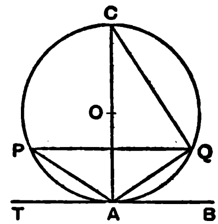


Fig. 655

[12] The tangent to a circle of radius 4·5 in. from an external point T is 6 in. long. Find the distance of T from the nearest point of the circumference.

13. The radii of two concentric circles are 5 cm., 3 cm. Find the length of a chord of the larger circle which touches the smaller circle.

Nos. 14–17 refer to fig. 656, in which a circle, centre I, touches the sides of △ABC at X, Y, Z.

14. If ∠B = 50°, ∠C = 70°, find the angles of △XYZ.

[15] If ∠XYZ = 64°, ∠XZY = 48°, find ∠XIZ and the angles of △ABC.

16. If AB = 8 cm., BC = 7 cm., CA = 5 cm., find BX.
[Let BX = BZ = $x$ cm., then CY = CX = $(7 - x)$ cm.; also AY = AZ.]

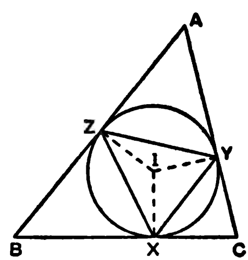


Fig. 656

[17] Repeat No. 16 if AB = 4 in., BC = 4·5 in., CA = 3·5 in.

18. In △ABC, ∠B = 50°, ∠C = 70°; a circle touches BC, AC produced, AB produced at P, Q, R respectively. Find ∠QPR.

[19] Three of the angles of a quadrilateral circumscribing a circle are 70°, 84°, 96° in order. Find the angles of the quadrilateral whose vertices are the points of contact.

20. In $\triangle ABC$, $BC=3$ cm., $CA=6$ cm., $AB=7$ cm.; a circle is drawn to touch $AB$ produced at $R$, $AC$ produced at $Q$, and $BC$ at $P$. Find the lengths of $AR$ and $CP$.

[21] Repeat No. 20 if $BC=3$ in., $CA=5$ in., $AB=4$ in.

22. In fig. 657, not drawn to scale, $QR$ touches each of the circles, centres $A$, $B$, radii 8 cm., 3 cm. respectively. If $AB=13$ cm., find $QR$.

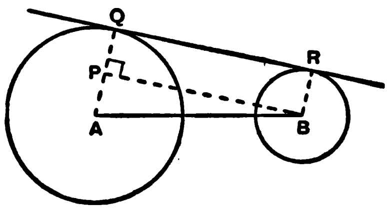



FIG. 657

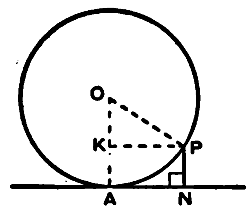



FIG. 658

*23. In fig. 658, $AN$ is a tangent to the circle and $ANP$ is a right angle. If $AN=15$ cm. and $PN=9$ cm., find the radius of the circle. [Draw $PK$ perpendicular to the radius $OA$; let $OA=r$ cm.]

*24. In $\triangle ABC$, $AB=2$ in., $BC=3$ in., $\angle ABC=90°$. Find the radius of the circle which touches $AB$ at $A$ and passes through $C$. [Draw $CK$ perpendicular to the radius $AO$ produced; join $OC$.]

*25. $AD$ is an altitude of $\triangle ABC$. If $AB=3$ in., $AC=4$ in., $AD=2{\cdot}4$ in., and if $D$ lies between $B$ and $C$, prove that $AC$ touches the circle, centre $B$, radius $BA$.

*26. A hemispherical bowl of diameter 25 in. and negligible thickness, rests on a horizontal table. Water is poured into the bowl till the surface of the water is $3\frac{1}{2}$ in. below the rim. If the bowl is tilted slowly, find the height above the table of the highest point of the rim when the water is about to overflow.

## THEOREM 58

**The straight line drawn perpendicular to a radius of a circle at its extremity is a tangent to the circle.**

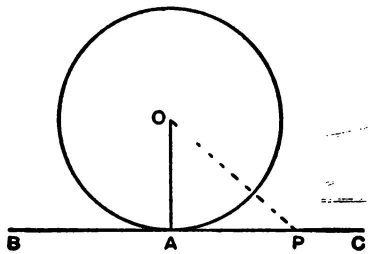



FIG. 659

**Given** a circle centre O, a radius OA, and the straight line BAC perpendicular to OA.

**To prove that** BAC is a tangent.

**Construction.** Take any point P on BC.
Join OP.

**Proof.** $\angle OAP = 1$ rt. $\angle$ *given*,

$\therefore$ each other angle of $\triangle OAP$ is less than 1 rt. $\angle$,

$\therefore$ $\angle OPA < \angle OAP$,

$\therefore$ $OA < OP$ *greater side opposite greater angle.*

But OA is a radius,

$\therefore$ OP is greater than a radius,

$\therefore$ P lies outside the circle.

Similarly, every point on BC except A lies outside the circle.

$\therefore$ BC touches the circle at A.

## THEOREM 59

**A tangent to a circle is perpendicular to the radius drawn through the point of contact.**

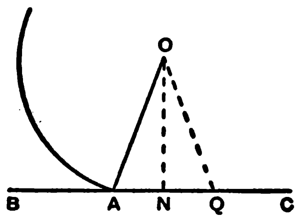



FIG. 660

**Given** a tangent BAC at a point A on a circle, centre O.

**To prove that** $\angle$OAC is a right angle.

**Construction and Proof.** *If possible*, suppose that OA is not perpendicular to BC and draw the perpendicular ON from O to BC.

Produce AN to Q so that AN = NQ.

Join OQ.

By construction, ON is the perpendicular bisector of AQ,

$\therefore$ OA = OQ *locus theorem*,

$\therefore$ Q lies on the circle,

$\therefore$ BAC cuts the circle at two points, A and Q.

But this is impossible because BAC is a tangent.
Therefore the original supposition is false.

$\therefore$ OA must be perpendicular to BC.

**Corollary 1. At every point of a circle, one and only one tangent can be drawn to the circle.**

**Corollary 2. The perpendicular to a tangent at its point of contact passes through the centre of the circle.**

## THEOREM 60

**If two tangents are drawn to a circle from an external point,**

**(1) the tangents are equal;**
**(2) the tangents subtend equal angles at the centre;**
**(3) the line joining the centre to the external point bisects the angle between the tangents.**

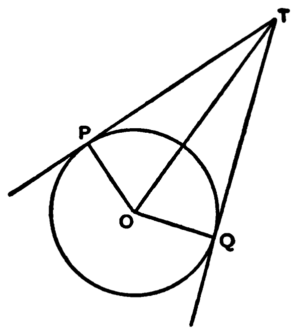



FIG. 661

**Given** a circle centre O and the tangents TP, TQ from an external point T to the circle.

**To prove that** (1) $TP = TQ$,
(2) $\angle TOP = \angle TOQ$,
(3) $\angle OTP = \angle OTQ$.

**Proof.** In $\triangle$s OPT, OQT,

$OP = OQ$ *radii*,
$OT = OT$,
$\angle OPT = \angle OQT$ *rt. $\angle$s, tangent perp. to radius*,

$\therefore$ $\triangle$s $\begin{matrix} OPT \\ OQT \end{matrix}$ are congruent RHS.

$\therefore$ $TP = TQ$,
and $\angle TOP = \angle TOQ$,
and $\angle OTP = \angle OTQ$.

### Examples for Oral Discussion

**1. PQ is a variable chord of a given circle, centre O. If PQ is of constant length, prove that it touches a fixed concentric circle.**

If ON is the perpendicular from O to PQ, explain why the length of ON is constant.

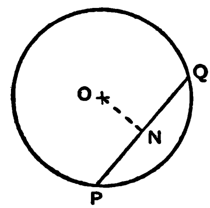


FIG. 662

**2. In fig. 663, ABCD is a quadrilateral circumscribing a circle. Prove that**

$$AB + CD = AD + BC.$$

Let the points of contact be P, Q, R, S. What do you know about AP? about BP?

**3. If, in fig. 663, O is the centre of the circle, prove that**

$$\angle AOB + \angle COD = 2 \text{ right angles.}$$

What do you know about $\angle OAB$? about $\angle OBA$?
What do you know about $\angle A + \angle B + \angle C + \angle D$?

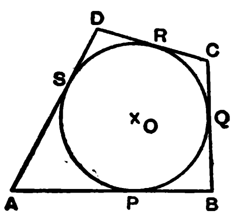


FIG. 663

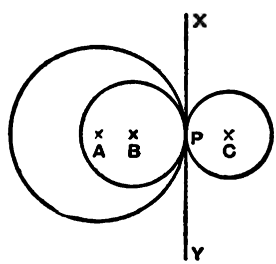


FIG. 664

**4. Fig. 664 represents three circles, centres A, B, C, touching a straight line XY at the point P. Prove that P, A, B, C are collinear.**

What do you know about $\angle CPX$?

## EXERCISE 66

1. AB is a diameter of the circle APB. Prove that the circle, centre A, radius AP, touches PB.

[2] AB is a diameter of a circle. Prove that the tangents at A and B are parallel.

3. AP is a chord of a circle, centre O. PN is the perpendicular from P to the tangent at A. Prove that AP bisects $\angle$OPN.

Nos. 4–7 refer to fig. 665 in which PQ, PR are chords of a circle, centre O, which touch a concentric circle at A, B.

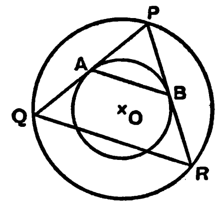



FIG. 665

4. Prove that PA = AQ.

[5] Prove that PQ = PR.

[6] Prove that QR = 2AB.

7. Prove that OP is the perpendicular bisector of AB.

8. AB is a diameter of the circle APB; AX, BY are the perpendiculars from A, B to the tangent at P. Prove that PX = PY. [Join P to the centre.]

Nos. 9–12 refer to fig. 656, p. 356, in which a circle touches the sides of $\triangle$ABC.

9. Prove that $\angle YXZ = 90° - \frac{1}{2}\angle BAC$. [Join Y, Z to the centre.]

[10] If Y, Z are the mid-points of AC, AB, prove that BX = CX.

11. Prove that $AY = \frac{1}{2}(AB + AC - BC)$.

*12. Prove that the inscribed circles of $\triangle$AXB, $\triangle$AXC touch AX at the same point.

13. ABCDEF is a hexagon circumscribing a circle. Prove that AB + CD + EF = BC + DE + FA.

14. P and Q are points on two circles which have the same centre O; the tangents at P and Q meet at T. Prove that (i) O, P, Q, T are concyclic; (ii) $\angle$OQP and $\angle$OTP are either equal or supplementary.

[15] AP, AQ are two chords of a circle and AB is a diameter. The tangent at B meets AP produced, AQ produced at X, Y. Prove that P, Q, X, Y are concyclic.

[16] If a parallelogram circumscribes a circle, prove that it is a rhombus and that the diagonals intersect at the centre of the circle.

17. In fig. 666, O is the centre of the circle, $\angle$AOT is a right angle, and TP is a tangent. Prove that TP = TQ. [Join OP.]

[18] O is the centre of the circle ABC. If $\angle$ABC = 30° and if the tangent at A cuts OC produced at T, prove that OC = CT.

19. AB is a diameter of a circle APB, centre O; the tangents at A and P meet at T. Prove that TO is parallel to PB. [Let AP cut OT at N; join OP.]

[20] O and A are fixed points; AP is the tangent from A to a variable circle, centre O. Prove that the locus of P is a circle.

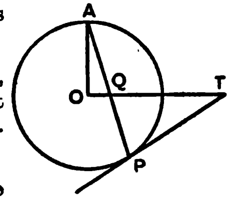


FIG. 666

[21] The tangents at points P, Q on a circle, centre O, meet at T; the line through T perpendicular to QT meets OP, produced if necessary, at K. Prove that KT = KO. [Join OQ, OT.]

22. Two parallel tangents to a circle, centre O, are cut by a third tangent at P, Q. Prove that $\angle$POQ is a right angle.

[23] In $\triangle$ABC, $\angle$ABC = 90°. A circle, centre X, is drawn to touch AB produced, AC produced, and BC. Prove that $\angle$AXC = 45°.

*24. PQ is a chord of a circle, centre O; the tangents at P, Q meet at T; TR is drawn perpendicular to TP and so that $\angle$QTR is acute. If $\angle$QTR > $\angle$OPQ, prove that $\angle$POQ > 120°.

*25. In fig. 667, APB is a semicircle, centre O, touching three sides of the rectangle ABCD; ANPC is a straight line and AN = NP. Prove that (i) O, B, C, N are concyclic; (ii) $\angle$ANB = 135°.

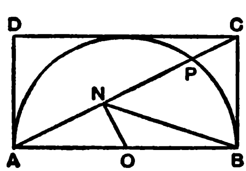


FIG. 667

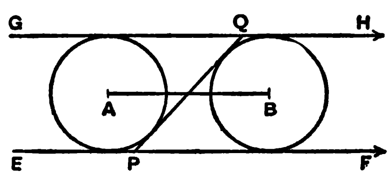


FIG. 668

*26. In fig. 668, A, B are the centres of two circles touching the parallel lines EPF, GQH; PQ touches each circle. Prove that (i) APBQ is a rectangle; (ii) PQ = AB.

*27. If in fig. 663, p. 361, BC, AD produced meet at H, and AB, DC produced meet at K, prove that HB + BK = HD + DK.

*28. PQRS is a minor arc of a circle; the tangents at P, S meet at B; the tangents at Q, R meet at D; RD produced cuts PB at A; QD produced cuts BS at C. Prove that AB − CD = BC − AD.

*29. A diameter AB of a circle APB is produced to C so that AB = 2BC. CT is the perpendicular from C to the tangent at P. Prove that (i) BP = BT; (ii) ∠PBT = 2∠ABP.

### Alternate Segment

**Examples for Oral Discussion**

Nos. 1, 2 refer to fig. 669, in which BAC is a tangent and AD is any chord.

1. If $\angle c = 52°$, find ∠APD.

Draw the diameter AE. Write down, *with reasons*, the sizes of $\angle e$, $\angle p_1$, $\angle p$.

**2. Prove that $\angle c = \angle p$.**

Draw the diameter AE. Explain why $\angle c + \angle e$ and $\angle p_1 + \angle e$ each equal 1 rt. ∠. Complete the proof.

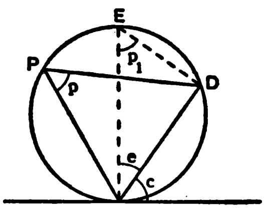



Fig. 669

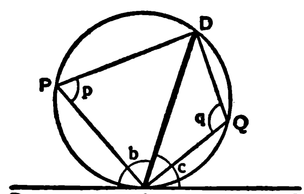



Fig. 670

Nos. 3, 4 refer to fig. 670, in which BAC is a tangent and AD is any chord.

3. If $\angle c = 52°$, find $\angle b$ and $\angle q$.

Use the fact, proved in No. 2, that $\angle c = \angle p$.

**4. Prove that $\angle b = \angle q$.**

Explain why $\angle b + \angle c$ and $\angle p + \angle q$ each equal 2 rt. ∠s.

Fig. 671 shows the facts proved in Nos. 2, 4.

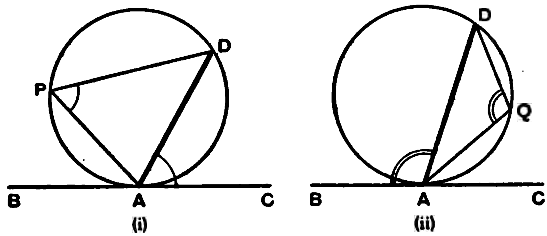



FIG. 671

If a straight line BAC touches a circle at A, and if AD is *any chord through the point of contact* A, $\angle$DAC and $\angle$DAB are the two angles formed by the chord AD with the tangent at A.

Since BAC is a straight line, these angles are supplementary.

In fig. 671 (i), $\angle$DAC is on the *right* of the chord AD, and it is equal to the angle in the segment cut off by AD on the *left* of AD.

In fig. 671 (ii), $\angle$DAB is on the *left* of the chord AD, and it is equal to the angle in the segment cut off by AD on the *right* of AD.

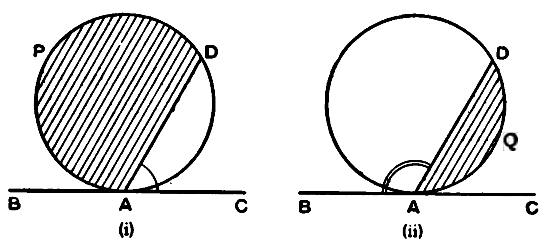



FIG. 672

The segment APD of the circle on the opposite side of AD to $\angle$DAC is called the **alternate segment** corresponding to $\angle$DAC. Similarly, the segment AQD of the circle on the opposite side of AD to $\angle$BAD is called the alternate segment corresponding to $\angle$BAD.

The results given in Nos. 2, 4 may therefore be stated as follows:—

**The angles which a tangent to a circle makes with any chord through the point of contact are equal to the angles in the alternate segments of the circle.**

5. Given a circle and a point A on it, inscribe a triangle ABC in the circle, such that $\angle B = 55°$, $\angle C = 75°$.

Draw the tangent SAT. Explain how AB and AC must be drawn.

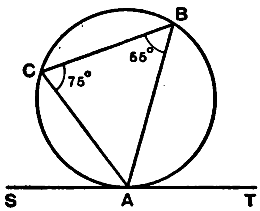


Fig. 673

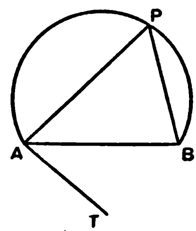


Fig. 674

6. Given a straight line AB, 2 in. long, construct a segment of a circle APB such that $\angle APB = 40°$. Measure the radius.

Draw AT so that $\angle BAT = 40°$. Explain how to find the centre of a circle which (i) touches AT at A and (ii) passes through A and B.

**7. If, in fig. 674, $\angle TAB = \angle APB$, prove that AT touches the circle APB at A.**

*If possible*, suppose AT is not a tangent and draw the tangent AK so that AK and AT are on the same side of AB. Explain why this is impossible.

Example 7 is the converse of Nos. 2, 4 and is often useful in rider work when it is necessary to prove that a line touches a circle.

# NUMERICAL EXAMPLES

## EXERCISE 67

Nos. 1–5 refer to fig. 675, in which SAT is the tangent at A to the circle ABC, and CBT is a straight line.

1. If $\angle$ CAT = 124°, find $\angle$ ABC.

[2] If $\angle$ TAC = 118°, $\angle$ ATC = 26°, find $\angle$ BAT.

3. If $\angle$ CAS = 65°, $\angle$ BAT = 47°, find the angles of $\triangle$ ABC.

[4] If $\angle$ BAS = 140° and $\angle$ ATC = 35°, find the angles of $\triangle$ ABC.

5. If $\angle$ ABC = 80° and $\angle$ ATC = 40°, prove that AC = AT.

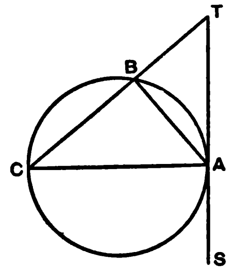


FIG. 675

[6] The inscribed circle of $\triangle$ ABC touches BC, CA, AB at X, Y, Z respectively. If $\angle$ B = 64°, $\angle$ C = 52°, find $\angle$ XYZ, $\angle$ XZY.

7. In fig. 676, AP and CQ are tangents. Find $\angle$ BCQ.

8. The tangents at A, B to the circle ABC meet at T, and AC is parallel to TB. If $\angle$ ATB = 54°, find the angles of $\triangle$ ABC.

[9] TBP, TCQ are tangents to the circle ABC. If $\angle$ PBA = 146°, $\angle$ QCA = 128°, find $\angle$ BAC, $\angle$ BTC.

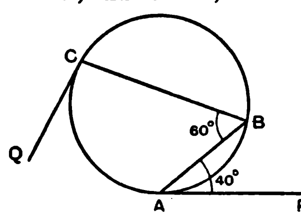


FIG. 676

10. The tangents at P, Q to the circle PQR meet at T. If $\angle$ PTQ = 36° and if the minor arc PR is three times the arc QR, find $\angle$ RPT. [Two answers.]

[11] In fig. 677, AP, DQ are tangents. If $\angle$ PAB = 42°, $\angle$ QDC = 55°, $\angle$ BDC = 24°, find the angles of the quadrilateral ABCD.

12. In fig. 677, AP, DQ are tangents and BD is a diameter. If $\angle$ BAP = $x$°, $\angle$ BPA = $y$°, find a relation between $x$ and $y$.

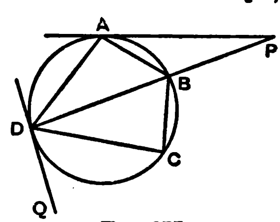


FIG. 677

[*Exercise* 67 *is continued on p.* 369.]

## THEOREM 61

**If a straight line touches a circle and from the point of contact a chord is drawn, the angles which the chord makes with the tangent are equal to the angles in the alternate segments of the circle.**

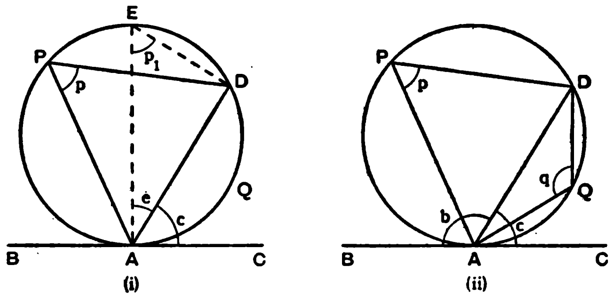



FIG. 680

**Given** a straight line BAC touching a circle at A and a chord AD forming the two segments APD, AQD.

**To prove** (1) $\angle$DAC $=$ $\angle$APD in alternate segment APD,
(2) $\angle$DAB $=$ $\angle$AQD in alternate segment AQD.

(1) **Construction.** Draw the diameter AE.
Join ED.

**Proof.** With the notation in fig. 680 (i),
since AE is a diameter and AC is a tangent,

$\angle c + \angle e = 1$ rt. $\angle$;

also $\angle$ADE $= 1$ rt. $\angle$ — *$\angle$ in semicircle,*

$\therefore \angle p_1 + \angle e = 1$ rt. $\angle$ — *$\angle$ sum of $\triangle$,*

$\therefore \angle c + \angle e = \angle p_1 + \angle e,$

$\therefore \angle c = \angle p_1.$

But $\angle p_1 = \angle p$ — *$\angle$s in same segment.*

$\therefore \angle c = \angle p.$

(2) **Proof.** With the notation in fig. 680 (ii),

$\angle b + \angle c = 2$ rt. $\angle$s *adj. $\angle$s on st. line,*

and $\angle q + \angle p = 2$ rt. $\angle$s *opp. $\angle$s cyclic quad.,*

$\therefore\ \angle b + \angle c = \angle q + \angle p.$

But $\angle c = \angle p$ *proved,*

$\therefore\ \angle b = \angle q.$

---

EXERCISE 67 (*continued*)

Nos. 13, 14 refer to fig. 678 in which QAPK, RBP are straight lines and PT is a tangent.

**13.** If $\angle$PQR = 75°, what other angles in the figure can be found? Find them.

[14] If $\angle$TPK = 72°, $\angle$TPR = 65°, find $\angle$PQR and $\angle$PRQ.

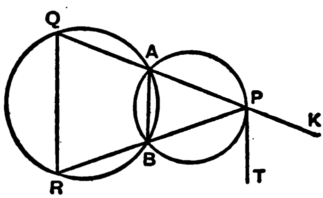



FIG. 678.

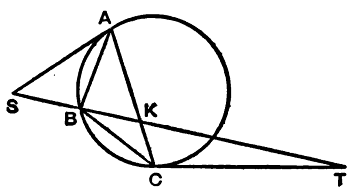



FIG. 679

**15.** In fig. 679, AS, CT are tangents and SBKT is a straight line. If $\angle$AST = 42°, $\angle$CKT = 55°, $\angle$SBC = 155°, find $\angle$BAC and $\angle$CTS.

**16.** In $\triangle$ABC, AB = AC and $\angle$B = 70°; BC is produced to D so that $\angle$CAD = 30°. Prove that AB touches the circle ADC.

[17] ABCD is a quadrilateral in which $\angle$BCD = 95°, $\angle$CDA = 60°. If $\angle$BAC = 55° and $\angle$ACD = 45°, prove that AD touches the circle ABC.

## THEOREM 62

**If a straight line be drawn from an extremity of a chord of a circle making with the chord an angle equal to the angle in the alternate segment, the straight line touches the circle.**

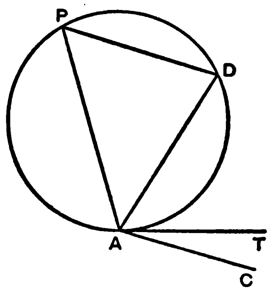



FIG. 681

**Given** a chord AD of a circle and a straight line AC such that $\angle$DAC = $\angle$APD in alternate segment APD.

**To prove that** AC touches the circle at A.

**Construction and Proof.** *If possible,* suppose that AC does not touch the circle at A and draw the tangent AT on the same side of AD as AC.

$\angle$TAD = $\angle$APD $\quad$ *$\angle$ in alt. segment,*

but $\angle$CAD = $\angle$APD $\quad$ *given,*

$\therefore$ $\angle$TAD = $\angle$CAD.

But this is impossible because AC and AT are on the same side of AD.

Therefore the original supposition is false.

$\therefore$ AC must touch the circle at A.

## EXERCISE 68

1. In fig. 682, AD is a tangent and CD is parallel to BA. Prove that ∠CDA = ∠ACB.

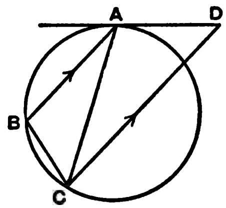



FIG. 682

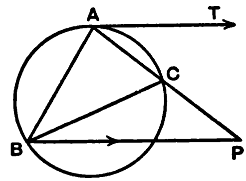



FIG. 683

2. In fig. 683, BP is parallel to the tangent AT; ACP is a straight line. Prove that ∠ABP = ∠ACB.

[3] ABC is a triangle inscribed in a circle; the tangents at B and C meet at T. If ∠BAC = 60°, prove that △TBC is equilateral.

4. A chord AC of a circle ABC is produced to T; TB is a tangent to the circle, see fig. 687. Prove that ∠TCB = ∠ABT.

[5] AQ, AP are the tangents at A to the circles ABP, ABQ, see fig. 686. Prove that ∠ABP = ∠ABQ.

[6] ABC is a △; the bisector of ∠BAC meets BC at D; a circle is drawn touching BC at D and passing through A. If the circle cuts AB, AC again at P, Q, prove that ∠PDB = ∠QDC.

7. In fig. 684, AP is a tangent and PBC is parallel to AD. If PQD is a straight line, prove that ∠PAQ = ∠BPQ.

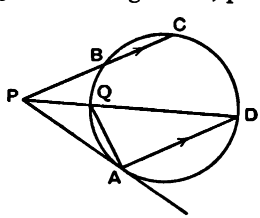



FIG. 684

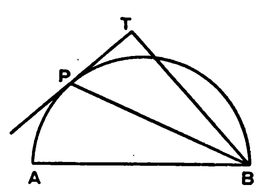



FIG. 685

8. In fig. 685, PT is a tangent to the semicircle APB. If BP bisects ∠ABT, prove that ∠BTP is a right angle. [Join AP.]

9. In fig. 686, AP, AQ are tangents and PHKQ is a straight line. Prove that AH = AK.

[10] If in fig. 686, in which AP, AQ are tangents, QB is produced to meet the circle ABP in R, prove that PR is parallel to AQ. [Join AB.]

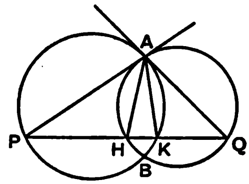


Fig. 686

[11] In $\triangle$ABC, $\angle$BAC is a right angle; D is any point on BC. If DP, DQ are the tangents at D to the circles ABD, ACD, prove that $\angle$PDQ is a right angle.

12. The tangent at B to a circle ABC meets a circle ABD at D, and the tangent at A to the circle ABD meets the circle ABC at C. Prove that AD is parallel to BC. [Join AB.]

[13] The tangent at B to the circle ABQ meets a circle ABP at T. If PAQ is a straight line, prove that BQ is parallel to TP.

14. In fig. 687, BT is a tangent and ACT is a straight line. If the bisector of $\angle$ABC cuts AC at P, prove that TP = TB.

[15] In fig. 687, BT is a tangent and ACT is a straight line. If $\angle$ABC = $\angle$T, prove that AB is a diameter.

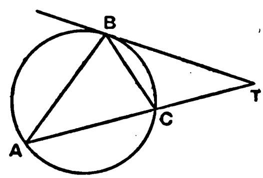


Fig. 687

[16] AB, AC are equal chords of a circle and such that CA produced meets at D the tangent BDE to the circle. Prove that $\angle$EDA = 3$\angle$EBA.

17. In $\triangle$ABC, AB = AC; a circle is drawn to touch BC at B and to pass through A; if it cuts AC again at D, prove that BC = BD.

18. A line AD is trisected at B, C; BPC is an equilateral triangle. Prove that AP touches the circle PBD.

[19] ABCDE is a regular pentagon; BD cuts CE at K. Prove that BC touches the circle BKE.

20. In fig. 688, AP, AQ are tangents and $\angle$PAQ is acute. Prove that $\angle$PBQ = 2$\angle$PAQ. What happens if $\angle$PAQ is obtuse?

[21] In fig. 688, AP, AQ are tangents. If P, B, Q are collinear, prove that AP, AQ are diameters of the circles ABP, ABQ.

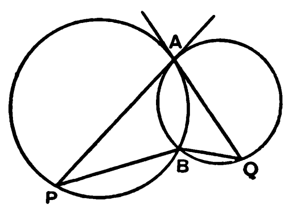


Fig. 688

[22] ABC is a triangle inscribed in a circle. Any line parallel to AC cuts BC at P and the tangent AQ at Q. Prove that A, B, P, Q are concyclic.

23. The straight line PQ touches the circles ABP, ABQ at P and Q. PA is produced to cut BQ at H; QA is produced to cut PB at K. Prove that A, H, B, K are concyclic.

[24] AB is a diameter of a circle ABC; TC is the tangent from a point T on AB produced; TD is drawn perpendicular to TA and meets AC produced at D. Prove that TC = TD.

25. ABCD is a cyclic quadrilateral; the line DE parallel to CB cuts AB, produced if necessary, at E. Prove that DC touches the circle DAE.

[26] ABCD is a cyclic quadrilateral such that the tangent at A to the circle is parallel to BD; AC cuts BD at E. Prove that (i) AC bisects $\angle$BCD, (ii) AB touches the circle CBE.

27. ABC is a minor arc of the circle ABCD. If the tangents at A and C meet at T, prove that $\angle$ATC = $\angle$ABC − $\angle$ADC.

[28] ABC, ABD are two equal circles. If AB = BC, prove that AC touches the circle ABD.

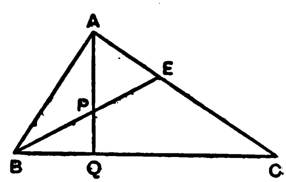


FIG. 689

29. In fig. 689, BPE bisects $\angle$ABC; APQ is a straight line such that AP = AE. Prove that AB touches the circle AQC.

[30] AB is a diameter of the circle AQRB, centre O. If AQ, BR meet when produced at P, prove that OQ, OR are tangents to the circle PQR. [Produce OQ to K.]

31. P is any point on the base BC of $\triangle$ABC. A circle is drawn to touch AB at B and to pass through P. If it cuts the circle ABC again at Q, prove that AC touches the circle CPQ. [Produce AC to K; join QB, QC, QP.]

[32] The bisector of $\angle$BAC cuts BC at D; a circle is drawn through D and to touch AC at A. Prove that its centre lies on the perpendicular from D to AB. [Let the circle cut AB at Q; join DQ.]

33. AB is a diameter of the circle ACB. If the tangents at A and C meet at T and if TC, AB are produced to meet at N, prove that $\angle BCN = \frac{1}{2}\angle ATN$.

34. In fig. 690, AS, AT are tangents to the circles APB, AQB, and APQ is a straight line. Prove that $\angle SAT = \angle PBQ$.

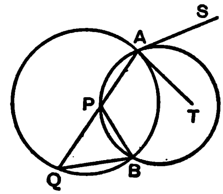


FIG. 690

*35. Two chords AOB, COD of a circle cut at O; the tangents at A and C meet at X; the tangents at B and D meet at Y. Prove that $\angle AXC + \angle BYD = 2\angle AOD$.

*36. The diameter AB of a circle APB, centre O, is produced to T so that OB = BT; TP is a tangent to the circle. Prove that TP = PA. [Draw BN parallel to OP to cut TP at N.]

*37. BAC, BAD are two circles such that the tangents at C and D meet at T on AB produced. If CBD is a straight line, prove (i) TCAD is a cyclic quadrilateral; (ii) $\angle TAC = \angle TAD$; (iii) TC = TD.

*38. ABCD is a minor arc of a circle such that AB = BC; AB and DC when produced meet at P, and DB is produced to meet the tangent AT at T. Prove that TP = TA. [Prove that TADP is a cyclic quadrilateral.]

*39. Assuming the result of Exercise 59, No. 19, p. 325, what special cases can be obtained by taking (i) P very close to R, (ii) P very close to A, (iii) A very close to B?

*40. OA is a chord of a circle, centre C; T is a point on the tangent at O such that OA = OT and $\angle AOT$ is acute; TA is produced to cut OC at B. Prove that $\angle OBA = \frac{1}{4}\angle OCA$. Find the position of B when A is very close to O.

*41. PQRS is a cyclic quadrilateral such that the sides PQ, QR, RS, SP touch a circle at A, B, C, D respectively. Prove that (i) AC is perpendicular to BD; (ii) the mid-points of AB, BC, CD, DA lie on a circle.

*42. PQ, CD are parallel chords of a circle; the tangent at D cuts PQ at T; B is the point of contact of the other tangent from T. Prove that BC bisects PQ. [If BC cuts PQ at K and if O is the centre, prove that (i) T, B, D, K are concyclic; (ii) T, B, D, O are concyclic.]

**Contact of Circles.** If two circles touch the same straight line at the same point, they are said to **touch each other** at that point. If the circles lie on opposite sides of the line, they are said to touch **externally**; if they lie on the same side of the line, they are said to touch **internally**.

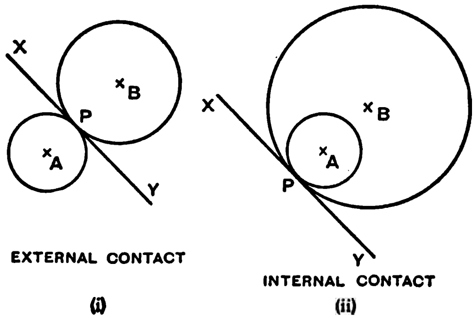



FIG. 691

## Examples for Oral Discussion

**1. In fig. 691, A and B are the centres of two circles which touch at P. Prove that A, B, P are collinear.**

What do you know about $\angle APX$ and about $\angle BPX$? Give reasons and complete the proof.

**2. If two circles touch externally, the distance between the centres is equal to the sum of the radii.**

**3. If two circles touch internally, the distance between the centres is equal to the difference of the radii.**

4. Fig. 692 represents three circles, centres A, B, C, radii 4 cm., 3 cm., 2·5 cm. respectively, touching one another. Find the lengths of the sides of $\triangle ABC$. Using instruments, draw $\triangle ABC$ and then draw the circles.

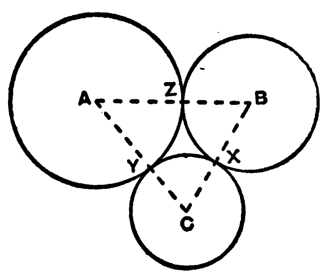



FIG. 692

5. Fig. 693 represents three circles, centres F, G, H, radii 2 in., 1·2 in., 0·6 in. respectively, touching one another. Which of the contacts are internal? Find the lengths of GH, HF, FG. Using instruments, draw △FGH and then draw the circles.

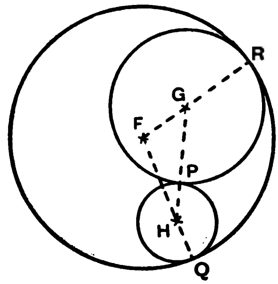


FIG. 693

6. If, in fig. 692, AB = 3·5 in., BC = 2·7 in., CA = 3·2 in., calculate the radii of the three circles and then draw the figure.

Let the radius of the circle, centre A, be $x$ in.; express the other radii in terms of $x$.

7. What is the *complete* locus of the centre of a variable circle of radius 3 cm. which touches a fixed circle of radius 5 cm.?

8. What is the *complete* locus of the centre of a variable circle of radius 7 cm. which touches a fixed circle of radius 4 cm.?

9. In fig. 694, C is a point on AB such that AC = 7 cm., CB = 3 cm. Find the radius of the circle, centre Q, which touches AB at C and also touches the circle, centre O, diameter AB.

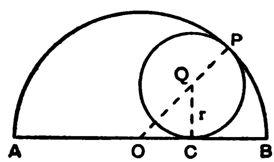


FIG. 694

If the radius is $r$ cm., explain why OQ = $(5 - r)$ cm.; and use the right-angled triangle OCQ.

10. In fig. 695, not drawn to scale, ABCD is a rectangle, AB = 4 cm., BC = 3 cm.; and A is the centre of a circle, radius 2 cm. Find the radii of the two circles which touch this circle and also touch BC at C.

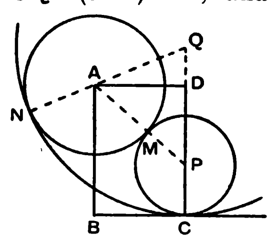


FIG. 695

Let P, Q be the centres of the required circles, radii $x$ cm., $y$ cm. respectively. Find in terms of $x$ the sides of △ADP; then use Pythagoras to find $x$. Similarly find $y$.

# NUMERICAL EXAMPLES

## EXERCISE 69

**1.** A circle, radius 5 cm., touches two concentric circles and encloses the smaller. The radius of the larger circle is 7 cm., find the radius of the smaller.

[2] The distance between the centres of two circles of radii 4 cm., 7 cm. is 15 cm. Find the radius of the least circle that can be drawn to touch them and enclose the smaller circle.

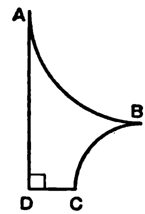



FIG. 696

**3.** In fig. 696, AB is a quadrant touching AD at A and the quadrant BC at B. If $\angle ADC = 90°$, $AD = 12$ in., $DC = 3$ in., find the radii of the circles.

**4.** Three circles, centres A, B, C, touch each other externally. If $AB = 4$ in., $BC = 6$ in., $CA = 7$ in., find the radii of the circles.

[5] Three circles, centres P, Q, R, touch each other externally. If $QR = 6$ cm., $RP = 7$ cm., $PQ = 8$ cm., find the radii of the circles.

**6.** Two circles, centres B, C, touch each other externally; a circle, centre A, touches the others internally. If $AB = 4$ in., $BC = 7$ in., $CA = 5$ in., find the radii of the circles.

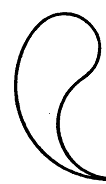


FIG. 697

**7.** Fig. 697 is formed of three circular arcs of radii 6·7 cm., 2·2 cm., 3·1 cm., touching one another; X, Y, Z are the centres of the circles. Find the lengths of the sides of $\triangle XYZ$.

**8.** State, without proof, the *complete* locus of the centre of a variable circle of radius 3·5 cm. which touches a fixed circle, centre A, of radius 2·3 cm.

**9.** Draw a line AB, 4 cm. long; draw a circle, centre A, radius 6 cm. and a circle, centre B, radius 2 cm. Construct a circle of radius 3 cm. to touch the larger circle internally and the smaller circle externally.

[10] Draw two circles, each of radius 2·5 cm., with their centres 6 cm. apart. Construct a circle of radius 7 cm. to touch one circle internally and the other externally.

**11.** Draw a circle, radius $1\frac{1}{2}$ in., and draw one of its diameters. Construct a circle of radius $\frac{1}{2}$ in. to touch the first circle and this diameter, and construct a circle of radius $\frac{1}{2}$ in. to touch the first circle and the diameter produced.

**12.** C is a point on AB such that AC $=5$ in., CB $=3$ in. Calculate the radius of the circle which touches AB at C and also touches the circle on AB as diameter.

[13] C is the mid-point of AB. Three semicircles are drawn on the same side of AB having AC, CB, AB as diameters. If AB $=12$ cm., calculate the radius of the circle which touches the three semicircles.

**14.** A, B are the centres of two circles of radii 5 cm., 3 cm.; AB $=12$ cm.; BC is a radius perpendicular to BA. Calculate the radius of a circle which touches the larger circle and also touches the smaller circle at C. [Two answers.]

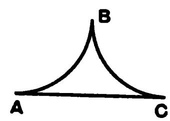


FIG. 698

**15.** In fig. 698, AB, BC are two equal quadrants touching at B; AC $=12$ cm.; find the radius of the circle which touches arc AB, arc BC, AC.

[16] ABCD is a square of side 7 in.; C is the centre of a circle of radius 3 in.; find the radius of each circle which touches this circle and touches AB at A.

***17.** Six circles each of radius 5 cm. are drawn with their centres at the vertices of a regular hexagon of side 10 cm. Find the radius of the circle which touches and encloses all of them.

***18.** In one corner of a square frame, side 3 ft., is placed a disc of radius 1 ft., touching both sides. Find the radius of the largest disc which will fit into the opposite corner.

***19.** OA $=a$ in., OB $=b$ in., $\angle$AOB $=90°$. Two variable circles are drawn touching each other externally, one of them touches OA at A, and the other touches OB at B. If their radii are $x$ in., $y$ in., prove that $(x+a)(y+b)$ is constant. If $a=8$, $b=6$, $x=4$, calculate $y$.

**Important Hint.** In solving problems about circles which touch each other, either internally or externally, it will often be found useful to draw the common tangent at their point of contact, as is done for proving Theorem 63. It is also useful, owing to Theorem 63, to join the centres to the point of contact.

## THEOREM 63

**If two circles touch one another, the line joining their centres (produced if necessary) passes through the point of contact.**

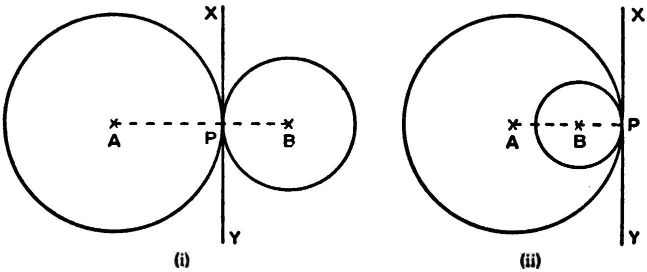



FIG. 699

**Given** two circles, centres A, B, touching each other at P.

**To prove that** A, B, P lie on a straight line.

**Construction and Proof.** Since the circles touch each other at P, they have a common tangent XPY at P.
Since the tangent to a circle is perpendicular to the radius through the point of contact,

$\angle APX$ and $\angle BPX$ are right angles.

$\therefore$ A and B lie on the line through P perpendicular to XPY;

$\therefore$ A, B, P lie on a straight line.

**Corollary 1.** **If two circles touch each other EXTERNALLY, the distance between the centres is equal to the SUM of the radii.**

In fig. 699 (i), $AB = AP + BP$.

**Corollary 2.** **If two circles touch each other INTERNALLY, the distance between the centres is equal to the DIFFERENCE of the radii.**

In fig. 699 (ii), $AB = AP - BP$.

## EXERCISE 70

Nos. 1–6 refer to fig. 700 in which the circles touch at A; the line HK touches the circles at H, K and meets at T the tangent at A.

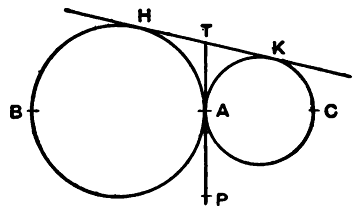


FIG. 700

1. Prove that TH = TK.

[2] If P is any point on AT, prove that the tangents from P to the circles are equal.

3. Prove that ∠HAK is a right angle.

[4] If AB is a diameter of the circle AHB, prove that AB touches the circle HAK.

5. If HA is produced to cut the circle AKC at M, prove that KM is a diameter of the circle AKC.

6. If any line RAS through A cuts the circles at R, S, prove that the tangents at R and S are parallel.

7. ABCD is a parallelogram. If the circles on AB and CD as diameters touch each other, prove that AB = BC. [Join the centres of the circles.]

8. ABCD is a parallelogram; AC cuts BD at K. Prove that the circles AKB, CKD touch each other. [Draw the tangent KT to the circle AKB.]

[9] ABCD is a straight line and P is a point such that ∠APB = ∠CPD. Prove that the circles PAD, PBC touch each other.

10. Two circles touch internally at A; a chord PQ of one touches the other at R. Prove that ∠PAR = ∠QAR. [Produce the tangent at A and the line PQ to meet at K.]

[11] Two circles touch internally at A; any line PQRS cuts one circle at P, S and the other at Q, R. Prove that

$$\angle PAQ = \angle RAS.$$

12. Two circles touch internally at A. The tangent at any point P on the inner circle cuts the outer at Q, R; AQ, AR cut the inner at H, K. Prove that the triangles PQH, APK are equiangular to one another.

[13] Two circles touch externally at A; a tangent to one of them at P cuts the other at QR. Prove $\angle PAQ + \angle PAR = 180°$. [Let the tangent at A cut PQ at T.]

14. T is any point on the tangent at A to the two circles AHP, AKQ which touch externally at A; TP, TQ are the other tangents from T to the circles and PHKQ is a straight line. Prove that (i) the tangents at P and K are parallel, (ii) $\angle PAH = \angle QAK$.

[15] Two circles touch externally at A; AB is a diameter of one circle; BR is the tangent from B to the other circle. Prove that $\angle ARB = 45° - \frac{1}{2}\angle ABR$.

16. Two equal circles touch externally at A; AB is a diameter of one circle; BR is the tangent from B to the other circle and cuts the first circle at Q. Prove that $BQ = \frac{2}{3}BR$. [Join AQ; join R to centre of second circle; use the intercept theorem.]

17. O is the centre of a fixed circle. Two variable circles, centres P, Q, touch the fixed circle internally and each other externally. Prove that the perimeter of $\triangle OPQ$ is constant.

[18] Four circular coins of unequal sizes lie on a table so that each touches two, and only two, of the others. Prove that the four points of contact are concyclic.

*19. Two circles, centres A, B, touch externally at P; a third circle, centre C, encloses both, touching the first at Q and the second at R. Prove that $\angle BAC = 2\angle PRQ$. [Draw tangent at R.]

*20. Two circles, centres B and C, touch externally at A; PQ is a line touching the circles at P and Q. Prove that the circle on BC as diameter touches PQ.

*21. C is the mid-point of AB; semicircles are drawn with AC, CB, AB as diameters and on the same side of AB. Prove that the radius of the circle which touches these three semicircles is equal to $\frac{1}{3}CA$.

*22. OA, OB are two radii of a circle such that $\angle AOB = 60°$; a circle touches OA, OB and the arc AB internally; prove that its radius is equal to $\frac{1}{3}OA$.

*23. In $\triangle ABC$, $AB = p$ in., $AC = q$ in., $\angle BAC = 90°$ and $p > q$; O is the mid-point of BC. Circles are drawn with AB and AC as diameters. Prove that two circles can be drawn with O as centre to touch each of these circles, and find their radii in terms of $p$, $q$.

*24. In fig. 701, APB is a semicircle and AQC is a quadrant, centre B; PQ is a tangent parallel to AB. Prove $\angle$PBQ = 15°.

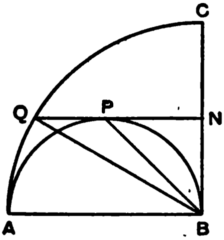



FIG. 701

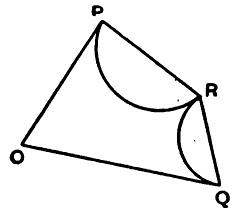



FIG. 702

*25. In fig. 702, PR, QR are two circular arcs touching each other at R and touching the unequal lines OP, OQ at P, Q. Prove that $\angle$PRQ = 180° − ½$\angle$POQ.

## CONSTRUCTIONS

The methods for performing simple tangent constructions have been given in previous exercises; for formal statements and proofs, see pp. 384–7.

**Definition.** If a straight line touches each of two circles, it is called a **common tangent** to the two circles; it is called a **direct** or **exterior** common tangent if the circles lie on the same side of it (fig. 705), and is called a **transverse** or **interior** common tangent if they lie on opposite sides of it (fig. 706).

## EXERCISE 71

1. Draw a circle, centre A, radius 3 cm., and take a point T 5 cm. from A. Construct the tangents from T to the circle, and calculate their lengths.

2. Draw a line AB, 4 cm. long. Construct a line AK such that its distance from B is 3 cm.

[3] Draw a line AB, 5 cm. long. Construct a circle, centre A, such that lengths of tangents from B to the circle are 3·5 cm.

4. Draw a circle, centre A, radius 4 cm., and a straight line BC at a distance 5 cm. from A. Construct a point T on BC such that the angle between the tangents from T to the circle is 80°.

**5.** Draw a circle, centre A, radius 4 cm., and take a point T 5 cm. from A. PQ is a variable chord of the circle of length 6 cm. Prove that PQ touches a fixed circle, centre A, and construct this circle. Construct a line through T cutting the given circle at H and K such that HK = 6 cm.

[6] Draw a circle, radius 3 cm., and take a point B 5 cm. from its centre. Construct a chord HK of the circle, 4 cm. long, such that HK produced passes through B.

**7.** Draw a circle, radius 4 cm., and take a point B 3 cm. from its centre. Construct a chord HK of the circle, 6·5 cm. long, such that HK passes through B.

[8] Draw a circle, centre A, radius 3 cm., and a straight line BC at a distance 5 cm. from A. Construct a point T on BC such that the length of the tangents from T to the circle is 4·5 cm.

[9] Two circles, radii 3 cm., 12 cm., touch each other externally. Calculate the length of their exterior common tangent (p. 386).

**10.** The distance between the centres of two circles, radii 11 cm., 5 cm., is 20 cm. Calculate the lengths of their exterior and interior common tangents (pp. 386, 387).

**11.** Draw two circles, radii 4 cm., 1·5 cm., with their centres 6 cm. apart. Construct an exterior common tangent. Measure its length and check by calculation.

**12.** Draw two circles, radii 3 cm., 2 cm., with their centres 7·5 cm. apart. Construct an interior common tangent. Measure its length and check by calculation.

[13] Draw a straight line AB 10 cm. long. Construct a line HK such that the lengths of the perpendiculars from A and B to HK are 3 cm., 4 cm. respectively. [Two cases.]

**14.** If the radii of two circles are $a$ in., $b$ in., and if the distance between their centres is $d$ in., where $d > a + b$, find in terms of $a$, $b$, $d$ the lengths of the common tangents.

[15] The diameters of two circles which touch externally are $a$ in., $b$ in., and the length of their exterior common tangent is $t$ in. Prove that $t^2 = ab$.

**16.** The distance between the centres of two circles is 10 cm., and the lengths of their exterior and interior common tangents are 8 cm., 6 cm. respectively. Calculate the radii of the circles.

## CONSTRUCTION 10

**Construct the tangent to a given circle at a given point on the circumference.**

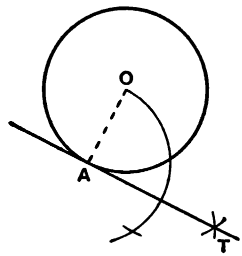



FIG. 703

**Given** a circle, centre O, and a point A on its circumference.

**To construct** the tangent at A to the circle.

**Construction.** Join AO.

Through A, construct the line AT perpendicular to AO.

Then AT is the required tangent.

**Proof.** The tangent is perpendicular to the radius through the point of contact.

But AO is a radius and $\angle$OAT is a right angle,

$\therefore$ AT is the tangent at A.

## CONSTRUCTION 11

**Construct the tangents to a given circle from a given point outside the circle.**

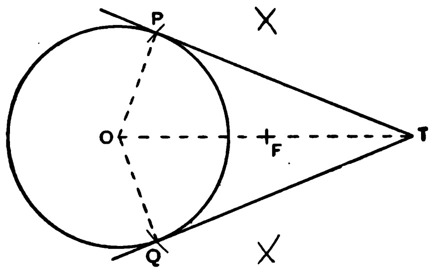



FIG. 704

**Given** a circle, centre O, and a point T outside the circle.

**To construct** the tangents from T to the circle.

**Construction.** Join OT and bisect it at F.

With centre F and radius FT, describe a circle and let it cut the given circle at P, Q.

Join TP, TQ.

Then TP, TQ are the required tangents.

**Proof.** Join OP, OQ.

Since TF = FO, the circle, centre F, radius FT, passes through O, and TO is a diameter.

$\therefore \angle TPO = 90^\circ$ $\quad$ *$\angle$ in semicircle,*

$\therefore$ PT is perpendicular to the radius through P,

$\therefore$ PT is the tangent at P to the given circle.

Similarly, it may be proved that QT is also a tangent to the given circle.

## CONSTRUCTION 12

**Construct the exterior common tangents to two circles.**

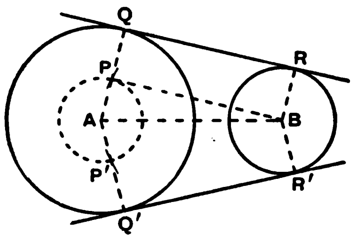



FIG. 705

**Given** two circles, centres A, B, radii $a$, $b$, and suppose $a > b$.

**To construct** the exterior common tangents to the two circles.

**Construction.** With centre A, radius $a - b$, describe a circle, and construct the tangents BP, BP′ from B to this circle.

Join AP, AP′ and produce them to cut the circle, radius $a$, at Q, Q′.

Through B, draw BR, BR′ parallel to AQ, AQ′ to meet the circle, centre B at R, R′.

Join QR, Q′R′.

Then QR, Q′R′ are the required common tangents.

**Proof.** By construction, AP $= a - b$ and AQ $= a$,

$\therefore$ PQ = AQ − AP $= b =$ BR,

$\therefore$ PQ and BR are equal and parallel,

$\therefore$ PQRB is a parallelogram.

But by construction PB is the tangent at P,

$\therefore$ $\angle$APB is a right angle,

$\therefore$ PQRB is a rectangle,

$\therefore$ $\angle$AQR and $\angle$BRQ are right angles,

$\therefore$ QR is perpendicular to the radii through Q and R.

$\therefore$ QR is a tangent to each circle.

Similarly, it may be proved that Q′R′ is a common tangent.

## CONSTRUCTION 13

**Construct the interior common tangents to two non-intersecting circles.**

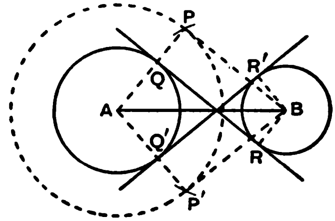



FIG. 706

**Given** two circles, centres A, B, radii $a$, $b$.

**To construct** the interior common tangents to the two circles.

**Construction.** With centre A, radius $a+b$, describe a circle, and construct the tangents BP, BP′ from B to this circle.

Join AP, AP′, cutting the circle, radius $a$, at Q, Q′.

Through B, draw BR, BR′ parallel to QA, Q′A to meet the circle, centre B, at R, R′.

Join QR, Q′R′.

Then QR, Q′R′ are the required common tangents.

**Proof.** By construction, $AP = a + b$ and $AQ = a$,

$\therefore$ $PQ = AP - AQ = b = BR$,

$\therefore$ QP and RB are equal and parallel,

$\therefore$ PQRB is a parallelogram.

But by construction PB is the tangent at P,

$\therefore$ $\angle APB$ is a right angle,

$\therefore$ PQRB is a rectangle,

$\therefore$ $\angle AQR$ and $\angle BRQ$ are right angles,

$\therefore$ QR is perpendicular to the radii through Q and R.

$\therefore$ QR is a tangent to each circle.

Similarly, it may be proved that Q′R′ is a common tangent.

## CONSTRUCTION 14

**Construct the inscribed circle of a given triangle.**

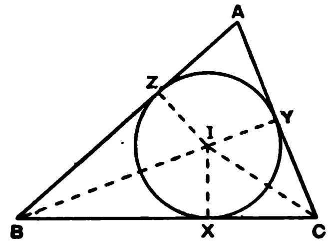



FIG. 707

**Given** a triangle ABC.

**To construct** the inscribed circle of the triangle ABC.

**Construction.** Construct the lines BI, CI bisecting the angles ABC, ACB and intersecting at I.
Draw IX perpendicular to BC.
With centre I, radius IX, describe a circle.
This is the required circle.

**Proof.** Draw IY, IZ perpendicular to AC, AB.
Since I lies on the line bisecting $\angle$ ABC,
I is equidistant from BA, BC,

$$\therefore \quad IZ = IX.$$

Similarly, $\qquad IY = IX.$

$\therefore$ the circle, centre I, radius IX, passes through X, Y, Z.
Also $\angle$IXC, $\angle$IYA, $\angle$IZA are right angles,

$\therefore$ BC, CA, AB are tangents to this circle.

NOTE. The definition of the inscribed circle of a triangle was given on p. 210, and the proof of Construction 14 is mainly a repetition of that of Theorem 33, p. 213. For the definition of the escribed circles of a triangle, see p. 211.

## CONSTRUCTION 15

**Construct an escribed circle of a given triangle.**

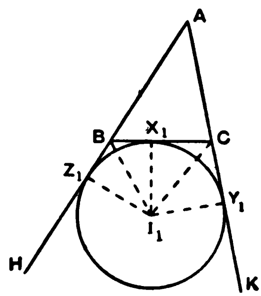



FIG. 708

**Given** a triangle ABC.

**To construct** the circle which touches AB produced, AC produced, and BC.

**Construction.** Produce AB, AC to H, K.
Construct the lines $BI_1$, $CI_1$ bisecting the angles HBC, KCB and intersecting at $I_1$.
Draw $I_1X_1$ perpendicular to BC.
With centre $I_1$, radius $I_1X_1$, describe a circle.
This is the required circle.

**Proof.** Draw $I_1Y_1$, $I_1Z_1$ perpendicular to ACK, ABH.
Since $I_1$ lies on the bisector of $\angle HBC$,
$I_1$ is equidistant from BH, BC,

$$\therefore\ I_1Z_1 = I_1X_1.$$

Similarly, $$I_1Y_1 = I_1X_1.$$

$\therefore$ the circle, centre $I_1$, radius $I_1X_1$, passes through $X_1$, $Y_1$, $Z_1$.
Also $\angle I_1X_1C$, $\angle I_1Y_1C$, $\angle I_1Z_1B$ are right angles,

$\therefore$ BC, ACK, ABH are tangents to this circle.

## CONSTRUCTION 16

**On a given straight line, construct a segment of a circle containing an angle equal to a given angle.**

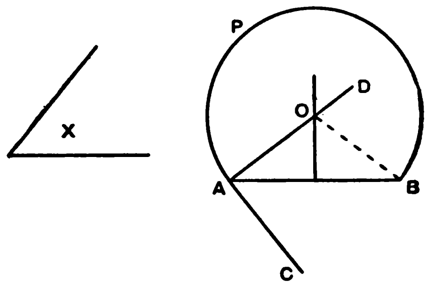



FIG. 709

**Given** a straight line AB and an angle X.

**To construct** a segment on AB containing an angle equal to $\angle$X.

**Construction.** At A make an angle BAC equal to $\angle$X.

Draw AD perpendicular to AC.

Draw the perpendicular bisector of AB and let it cut AD at O.

With O as centre and OA as radius describe a circle.

Then the segment of this circle on the side of AB opposite to C is the required segment.

**Proof.** Join OB.

Since O lies on the perpendicular bisector of AB,

$$OA = OB.$$

$\therefore$ the circle passes through B.

Since AC is perpendicular to the radius through A, AC is a tangent,

$\therefore$ $\angle X = \angle CAB$

$\quad = $ angle in alternate segment APB,

$\therefore$ the segment APB is the required segment.

A shorter method can be used for Construction 16 if the use of a protractor is allowed.

Suppose, for example, with the data of fig. 709, $\angle X = 38°$.

Then $$\angle OAB = 90° - 38° = 52°.$$

Also $$OA = OB,$$

$$\therefore \angle OBA = \angle OAB = 52°.$$

Therefore the centre O of the required segment is found by drawing an isosceles triangle OAB, with base AB and each base angle equal to 52°.

## Inscribed and Circumscribed Regular Polygons

If a regular polygon of $n$ sides is inscribed in a circle or circumscribed about a circle, each side subtends an angle of $\frac{360}{n}$ degrees at the centre of the circle.

For the values $n = 3, 4, 6, 8$, these angles are respectively 120°, 90°, 60°, 45°, and it has already been shown that angles of these magnitudes can be constructed without using a protractor.

Fig. 710 represents a regular octagon inscribed in a circle and one circumscribed about a circle.

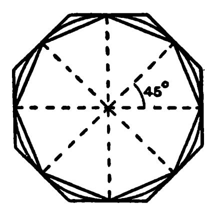



FIG. 710

The reader should construct inscribed and circumscribed regular figures of 3, 4, 6 sides, and, using a protractor, regular polygons of 5 sides and 7 sides.

The simplest way of inscribing a regular hexagon in a circle is to make use of the fact that the length of each side is equal to the radius.

To circumscribe a regular hexagon about a circle, draw tangents at the corners of the inscribed regular hexagon.

Alternate vertices of a regular hexagon are the vertices of an equilateral triangle.

### Examples for Oral Discussion

**1. In a given circle, inscribe a triangle equiangular to a given triangle PQR.**

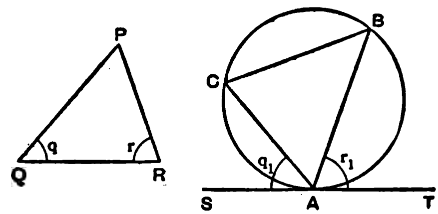


FIG. 711

Draw the tangent SAT at any point A on the given circle.
Draw AC, AB so that $q_1 = q$ and $r_1 = r$, and prove that $\triangle$ABC is equiangular to $\triangle$PQR.

**2. About a given circle, circumscribe a triangle equiangular to a given triangle PQR.**

Let I be the centre of the given circle; draw radii IX, IY, IZ so that $m_1 = m$ and $n_1 = n$.

Draw the tangents at X, Y, Z to form $\triangle$ABC, and prove that $\triangle$ABC is equiangular to $\triangle$PQR.

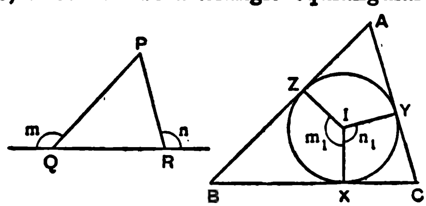


FIG. 712

**3. Construct a circle to pass through a given point A and to touch a given circle, centre O, at a given point B.**

Draw the perpendicular bisector LM of AB, and let it cut OB, produced if necessary, at P.

With centre P, radius PB, describe a circle, and prove that this is the required circle.

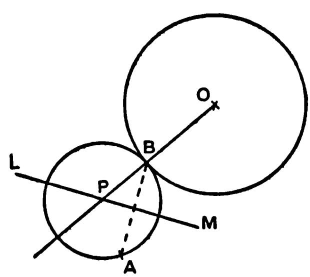


FIG. 713

4. Construct a triangle ABC, such that AB = 6 cm., $\angle$ACB = 50°, and the median CX = 5·5 cm.

Draw AB and bisect it at X.

Draw on AB a segment APB containing an angle of 50°.

Draw a circle, centre X, radius 5·5 cm.; this cuts the arc APB at C (two positions).

**5. Construct a circle to touch a given line AB and a given circle, centre C, at a given point D.** (*Two answers.*)

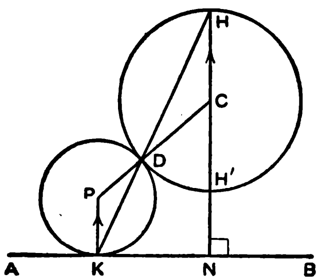



FIG. 714

Draw CN perpendicular to AB and let it cut the given circle at H, H′.

(i) HD produced cuts AB at K.
Draw KP parallel to NCH and let it meet CD produced at P.
Then the circle, centre P, radius PD, is the required circle.
Explain why PK = PD and complete the proof.

(ii) By using H′ instead of H in (i), obtain a second circle touching AB and the given circle at D (internally).

**6. Construct a circle to touch a given line AB at a given point K and a given circle, centre C.** (*Two answers.*)

Use the construction indicated in fig. 714. State the method and prove that it is correct.

**7. Construct a circle to touch a given circle, centre A, at a given point B and a second given circle, centre C.** ***(Two answers.)***

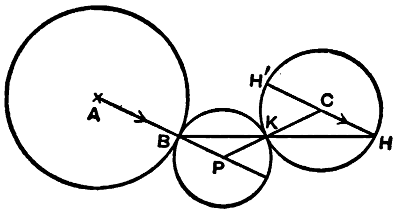



FIG. 715

Draw the diameter H′CH parallel to AB.

(i) Join BH and let it cut the circle, centre C, again at K. Produce AB and CK to meet at P.
Then the circle, centre P, radius PB, is the required circle.
Explain why PB = PK and complete the proof.

(ii) By using H′ instead of H in (i), obtain a second circle touching the circle, centre A, at B and touching the circle, centre C.

## EXERCISE 72

**1.** Draw $\angle BAC = 35°$ and make $AB = 4$ cm. Construct a circle to touch AC at A and pass through B. Measure its radius.

[2] Draw a line AB and take a point C 3 cm. from AB. Construct two circles of radius 4 cm. to pass through C and touch AB.

[3] Draw a circle of radius 4 cm. and inscribe an equilateral triangle in the circle. Measure its side.

**4.** Draw a circle of radius 2 cm. and circumscribe an equilateral triangle about the circle. Measure its side.

[5] Draw $\angle BAC = 65°$. Construct a circle of radius 3 cm. to touch AB and AC.

**6.** Take two points A, B, 4 cm. apart. Construct a circle to pass through A and B and such that the tangents at A and B include an angle of 100°. Measure its radius.

7. Draw a circle, radius 3 cm., and take a point A 4 cm. from its centre. Construct a circle of radius 2 cm. to touch this circle and to pass through A.

8. Take a point C 4 cm. from a line AB and draw a circle, centre C, radius 3 cm. Construct a circle of radius 2 cm. to touch this circle and AB.

[9] The centre C of a given circle, radius $a$ cm., is $d$ cm. from a given line AB. Under what conditions is it impossible to draw a circle of radius $b$ cm. to touch this circle and AB?

10. Take two points A, B, 6 cm. apart. With A, B as centres and radii 3 cm., 2 cm. respectively, describe two circles. Construct a circle of radius 5 cm. to touch each of these circles (internally or externally). Give all possible answers.

[11] Draw $\triangle$ ABC such that AB = AC = 7 cm., BC = 5 cm. Draw a circle, centre A, radius 3 cm. Construct a circle to touch this circle and pass through B and C.

12. Draw a circle, radius 4·5 cm., and a diameter AB. Construct a circle of radius 1·5 cm. to touch AB (or AB produced) and to touch the given circle (i) internally, (ii) externally.

[13] Draw a circle, radius 5 cm. Construct two circles, radii 1·5 cm., 2·5 cm. touching each other externally and touching the given circle internally.

14. Draw $\triangle$ ABC such that BC = 4 cm., CA = 3 cm., AB = 2 cm. Construct the four circles which touch the sides of $\triangle$ ABC and measure their radii.

[15] Draw $\triangle$ ABC such that AB = 4 cm., BC = 6 cm., $\angle$ B = 90°. Construct the circle escribed to BC and measure its radius.

16. Draw a quadrilateral such that its sides in order are 4, 5, 7, 6 cm. Inscribe a circle in it to touch three of the sides. Does it touch the fourth side?

17. Draw $\angle$ AOB = 40° and make OA = 4 cm. Construct a circle touching OA at A and touching OB. Measure its radius.

[18] Draw $\triangle$ ABC such that BC = 7 cm., CA = 6 cm., AB = 5 cm. Construct a circle to touch AB and AC and to have its centre on BC. Measure its radius.

19. Draw two parallel lines AB, CD, 6 cm. apart and take a point E between them, 2 cm. from AB. Construct a circle to touch AB and CD and to pass through E.

[20] Draw two parallel lines AB, CD and any circle cutting AB. Construct a circle to touch AB, CD and the given circle.

[21] Given two points A, B and a point D on a line CDE, show how to construct two concentric circles one of which passes through A and B, and the other touches CE at D. When is this impossible?

Construct the figures in Nos. 22–29. Arcs which meet are tangential where they meet unless otherwise indicated.

**Do not rub out any of the construction lines.**

**22.** Fig. 716 shows three arcs each of radius 3 cm. and each $\frac{1}{6}$th of a complete circumference. The arcs are not tangential at B or C.

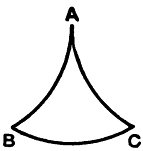



FIG. 716

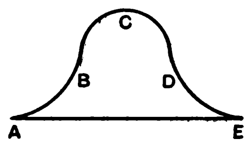



FIG. 717

[23] Fig. 717 shows four equal quadrants AB, BC, CD, DE; AE = 6 cm.

[24] Fig. 718 shows three arcs, each of radius 3 cm.

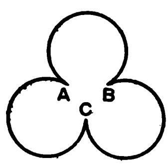



FIG. 718

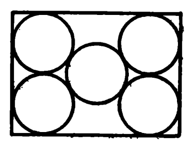


FIG. 719

**25.** Fig. 719 shows a rectangle 6 cm. by 8 cm. and the four outer circles are equal

**26.** In fig. 720, the radii of the arcs AB, BC, CA are 3·5 cm., 2·5 cm., 7 cm.

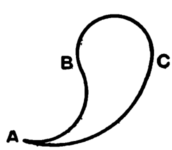



FIG. 720

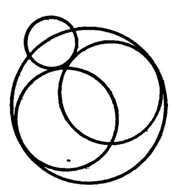


FIG. 721

[27] In fig. 721, the radii of the circles are 1 cm., 2 cm., 2 cm., 3 cm., and the centre of the smallest circle lies on the largest.

28. In fig. 722, AP, AQ are arcs of radii 4 cm., not tangential at A; PQ is an arc of radius 8 cm.; AB is perpendicular to CD and equals 3 cm.

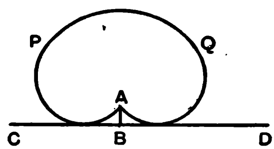



FIG. 722

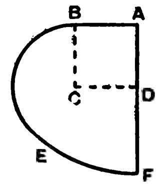



FIG. 723

[29] In fig. 723, ABCD is a square of side 2 cm.; BE, EF are arcs with C, A as centres respectively.

[30] On a line of length 5 cm., construct a segment of a circle containing an angle of 70°. Measure its radius.

31. On a line of length 2 in., construct a segment of a circle containing an angle of 140°. Measure its radius.

32. In a circle of radius 3 cm., inscribe a triangle whose angles are 40°, 65°, 75°. Measure its longest side.

33. Circumscribe about a circle of radius 2 cm. a triangle whose angles are 50°, 55°, 75°. Measure its longest side.

[34] Construct $\triangle ABC$ such that $BC = 6$ cm., $\angle BAC = 90°$, altitude $AD = 2$ cm. Measure AB, AC.

35. Construct $\triangle ABC$ such that $BC = 5$ cm., $\angle BAC = 55°$, altitude $AD = 4$ cm. Measure AB, AC.

36. Construct $\triangle ABC$ such that $BC = 6$ cm., $\angle BAC = 52°$, median $AX = 5$ cm.

[37] Draw $\triangle ABC$ such that $AB = 1{\cdot}8$ in., $BC = 2{\cdot}6$ in., $\angle ABC = 130°$. Find a point P within $\angle ABC$ such that $\angle BPC = 50°$ and area of $\triangle PAB = 2{\cdot}7$ sq. in.

38. Construct the quadrilateral ABCD such that $AD = 5$ cm., $BC = 4{\cdot}6$ cm., $\angle ABD = \angle ACD = 55°$, $\angle CBD = 43°$ Measure CD.

[39] The vertices of the pentagon ABCDE are concyclic; $AB = 2$ in., $BC = 3$ in., $\angle ADB = 30°$, area of $\triangle ABC$ = area of $\triangle ABE$, and BD bisects $\angle CBE$. Construct the pentagon.

40. A is a point on a circle of radius 5 cm.; P is a point on the tangent at A such that $AP = 8$ cm. Construct a circle to touch the given circle and touch AP at P. Measure its radius.

41. Draw $\angle ABC = 55°$ and make $BC = 7$ cm.; draw a circle, centre C, radius 3 cm., and let it cut CB at X. Construct a circle to touch the given circle externally at X and to touch AB. Measure its radius.

Construct the figures in Nos 42–45. Arcs which meet are tangential where they meet.

**Do not rub out any of the construction lines.**

42. In fig. 724, AB, AD are arcs of radii 6 cm.; the distance of C from A is 6 cm. and the line AC is an axis of symmetry.

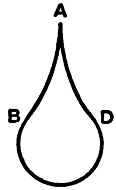


FIG. 724

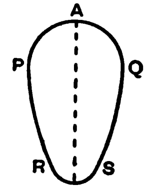


FIG. 725

43. In fig. 725, AB is an axis of symmetry; PAQ is a semicircle of radius 2 cm.; RBS is an arc of radius 1 cm.; $AB = 7$ cm.

[44] In fig. 726, AB is a quadrant of radius 2·5 cm. with its centre on AC; $AC = 7$ cm.

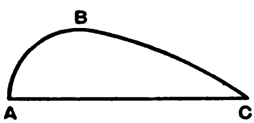


FIG. 726

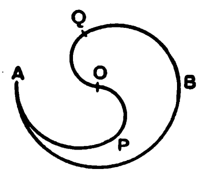


FIG. 727

45. In fig. 727, AB is a semicircle, radius 3 cm., centre O; OP, OQ are arcs each of radius 1 cm.

[46] ABC is an equilateral triangle; $AB = 4$ cm.; A, B are the centres of two circles, each of radius 2·5 cm.; CA is produced to meet the first circle at D. Construct a circle touching the first circle internally at D and touching the second circle externally.

47. Draw a circle centre O and two radii OA, OB. Show how to inscribe a circle in the sector AOB, *i.e.* to touch OA, OB, and arc AB.

*48. Draw any triangle ABC. Without making any measurements, construct three circles, centres A, B, C, so that each touches the other two. Give the four possible answers.

## Examples for Oral Discussion

**The In-circle and Ex-circles of a Triangle.** Nos. 1–4 refer to fig. 728, in which I is the in-centre and $I_1$ is an ex-centre of $\triangle ABC$ (see pp. 388, 389).

$BC = a$, $CA = b$, $AB = c$, $s = \frac{1}{2}(a+b+c)$,
area of triangle $ABC = \Delta$.

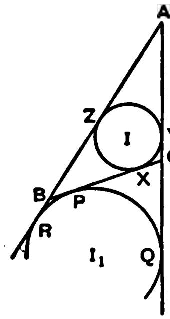



FIG. 728

1. Prove that $AY = s - a$. Write down corresponding expressions for BX, CX.

2. Prove that $AR = s$, and find BP, PC in terms of $a$, $b$, $c$, $s$.

3. Prove that the radius $r$ of the in-circle equals $\dfrac{\Delta}{s}$.

[$\triangle IBC + \triangle ICA + \triangle IAB = \triangle ABC$;
$\triangle IBC = \frac{1}{2}r \,.\, a$, etc.]

4. Prove that the radius $r_1$ of the ex-circle, centre $I_1$, equals $\dfrac{\Delta}{s-a}$.

[$\triangle I_1AB + \triangle I_1AC - \triangle I_1BC = \triangle ABC$.]

**The Orthocentre of a Triangle.** Nos. 5–10 refer to fig. 729, in which H is the orthocentre of $\triangle ABC$ (see p. 203).

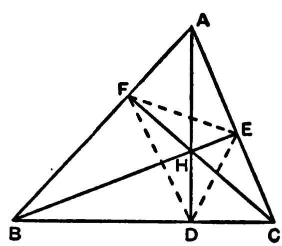



FIG. 729

5. Where is the orthocentre of $\triangle HBC$?

6. What can you say about the circles whose diameters are (i) AH, (ii) BC?

7. Prove that $\angle BHC = 180° - \angle BAC$.

8. Prove that $\triangle AEF$ is equiangular to $\triangle ABC$.

9. Prove that $\angle HDE = \angle HDF = 90° - \angle BAC$.

**10.** **(Nine-point Circle)** If AD, BE, CF are the altitudes of $\triangle$ABC, if H is the orthocentre, and if X, Y, Z, P, Q, R are the mid-points of BC, CA, AB, HA, HB, HC, respectively, prove that

(i) PZ is parallel to BE, and ZX is parallel to AC;
(ii) $\angle$PZX $=90°$ and $\angle$PYX $=90°$;
(iii) P, Z, X, D, Y lie on a circle;
(iv) the circle through X, Y, Z passes through P, Q, R, D, E, F (this circle is called the nine-point circle of $\triangle$ABC);
(v) the radius of the nine-point circle of $\triangle$ABC is half the circumradius of $\triangle$ABC. [YZ $=\frac{1}{2}$BC, etc.]

## Loci

**11.** If the base BC of $\triangle$ABC is given in magnitude and position and if $\angle$BAC is given in magnitude, find the locus of the orthocentre H of $\triangle$ABC.

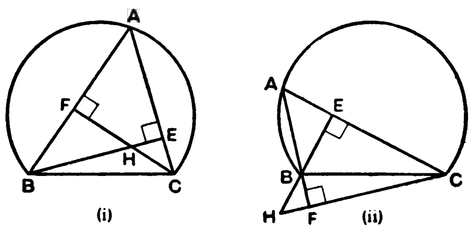


FIG. 730

(i) Prove that $\angle$BHC $=180° - \angle$BAC in fig. 730 (i) where $\angle$ABC and $\angle$ACB are acute and that $\angle$BHC $=\angle$BAC in fig. 730 (ii) where $\angle$ABC is obtuse.
(ii) What is the position of H when $\angle$ABC is a right angle?
(iii) What can you say about the path traced out by H when $\angle$ABC, $\angle$ACB are acute? What can you say if one of them is obtuse? Are these paths part of the same circle?
(iv) What is the position of H when A is very close to B?
(v) What is the *complete* locus of H?
(vi) Draw the locus of H if BC $=3$ cm. and $\angle$BAC $=30°$.

**12.** If the base BC of $\triangle$ABC is given in magnitude and position and if $\angle$BAC is given in magnitude, find the locus of the in-centre I of $\triangle$ABC.

(i) What is the *complete* locus of A?

(ii) Prove that $\angle$BIC $= 90° + \frac{1}{2}\angle$BAC.

(iii) What is the position of I when A is very close to B?

(iv) What is the *complete* locus of I?

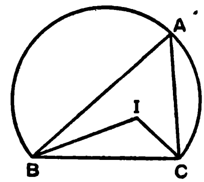



FIG. 731

## EXERCISE 73

Nos. 1–9 refer to fig. 728, p. 399, representing the in-circle and one ex-circle of $\triangle$ ABC.

**1.** Prove that (i) YQ = ZR; (ii) BP = XC.

[2] If AB > AC, prove that PX = AB − AC.

**3.** Prove that B, I, C, $I_1$ are concyclic.

[4] Prove that $\angle$AIC $= 90° + \frac{1}{2}\angle$ABC.

**5.** If $\angle$BIC = 100°, find $\angle$BAC.

[6] Prove that AZ + BX + CY $= \frac{1}{2}$(BC + CA + AB).

[7] Prove that AB − AC = BX − XC.

**8.** If AD is an altitude of $\triangle$ ABC and if AB > AC, prove that $\angle$IAD $= \frac{1}{2}(\angle$ACB $- \angle$ABC).

**9.** If AD is an altitude of $\triangle$ ABC and if O is the circumcentre, prove that AI bisects $\angle$OAD.

[10] AB is a chord of a circle; the tangents at A, B meet at T. Prove that the in-centre of $\triangle$ TAB lies on the circle.

Nos. 11–14 refer to fig. 729, p. 399, representing a triangle ABC and its orthocentre H.

**11.** Prove that $\triangle$s BDF, EDC are equiangular.

[12] If O is the circumcentre of $\triangle$ ABC, prove $\angle$HBA = $\angle$OBC.

**13.** Prove that H is the in-centre of $\triangle$DEF. What points are the ex-centres of $\triangle$DEF?

**14.** Prove that the circumcircles of $\triangle$ AHB, $\triangle$ AHC are equal.

[15] If I is the in-centre and $I_1$, $I_2$, $I_3$ are the ex-centres of $\triangle$ ABC, prove that $I_1$ is the orthocentre of $\triangle II_2I_3$.

16. If $I_1$, $I_2$, $I_3$ are the ex-centres of $\triangle$ ABC, what is the nine-point circle of $\triangle I_1I_2I_3$ and what follows from this fact?

17. In $\triangle$ ABC, $\angle BAC = 90°$. Prove that the diameter of the in-circle of $\triangle$ ABC equals $AB + AC - BC$.

[18] If the in-circle of $\triangle$ ABC touches BC at X, prove that the in-circles of $\triangle$ ABX, $\triangle$ ACX touch each other.

19. ABCD is a quadrilateral circumscribing a circle. Prove that the in-circles of $\triangle$ ABC, $\triangle$ ADC touch each other.

20. If in fig. 728, p. 399, $II_1$ cuts the circumcircle of $\triangle$ ABC at K, prove that I, $I_1$, B, C lie on a circle, centre K.

21. If in fig. 729, p. 399, AD produced cuts the circumcircle of $\triangle$ ABC at P, prove that $HD = DP$.

[22] If in fig. 729, p. 399, BH produced cuts the circumcircle of $\triangle$ ABC at K, prove that $AH = AK$.

[23] If in fig. 728, p. 399, the circumcircle of $\triangle$ BIC cuts AB at M, prove that $AM = AC$.

*24. If in fig. 729, p. 399, O is the circumcentre of $\triangle$ ABC, prove that OA is perpendicular to EF.

*25. If in fig. 728, p. 399, AM, AN are the perpendiculars from A to BI, CI, prove that MN is parallel to BC.

*26. H is the orthocentre and O is the circumcentre of $\triangle$ ABC; AK is a diameter of the circumcircle. Prove that

(i) BHCK is a parallelogram,
(ii) CH equals twice the distance of O from AB.

## LOCI

27. A is a fixed point on a fixed circle, centre O; AP is a variable chord of the circle. Prove that the locus of the mid-point of AP is the circle on OA as diameter.

[28] A variable chord PQ of a given circle pass through a fixed point. Find the locus of the mid-point of PQ.

29. A and B are fixed points; AP is the tangent from A to a variable circle, centre B. Find the locus of P.

30. A and B are fixed points; ABPQ is a variable parallelogram; the bisectors of $\angle$QAB, $\angle$PBA meet at R. Find the locus of R.

[31] The extremities of a line PQ of given length move along two given perpendicular lines OA, OB. Prove that the locus of the mid-point of PQ is a circle, centre O.

32. A and B are fixed points; P is a variable point on the given circle ABP; AP is produced to Q so that PQ = PB. Find the *complete* locus of Q.

[33] If the base BC of $\triangle$ ABC is given in magnitude and position and if $\angle$BAC is given in magnitude, find the *complete* locus of $I_1$, the centre of the ex-circle escribed to BC.

34. With the data of No. 33 find the *complete* locus of $I_2$, the centre of the ex-circle escribed to AC. [$\angle BI_2C = \frac{1}{2}\angle BAC$.]

35. Two given circles cut at A, B; P is a variable point on one of the circles; PA, PB, produced if necessary, cut the other circle again at Q, R. If AR and BQ, produced if necessary, meet at S, find the locus of S.

[36] A and B are fixed points on a given circle; PQ is a variable chord of the circle of given length; AP, BQ, produced if necessary, meet at R. Find the *complete* locus of R.

*37. In fig. 732, ABC is a given triangle. If $\angle$PAQ = $\angle$BAC and $\angle$PQA = $\angle$BCA, find the locus of P. [Join PB.]

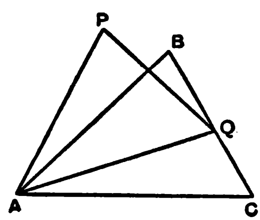


FIG. 732

*38. P, Q are variable points on the fixed lines AB, AC (not produced in the senses BA, CA); the perpendiculars at P, Q to AB, AC respectively meet at R. If PQ is of given length, prove that the locus of R is part of a circle, centre A, and state what part of this circle belongs to the locus.

*39. ABC is a given triangle such that AB = AC; P is a variable point such that $\angle$APB = $\angle$APC. Prove that the locus of P consists of one complete straight line, part of a second straight line, and part of the circle ABC. State the precise locus.

*40. PQ is a variable chord of given length of a given circle; A is a fixed point on the circle. Prove that the locus of the orthocentre of $\triangle$ APQ is a circle, centre A.

## MISCELLANEOUS CONSTRUCTIONS (*Revision*)

### EXERCISE 74

1. Use a coin to draw a circle and construct its centre

[2] Draw a line AB, 3 cm. long. Construct a circle, radius 5 cm., to pass through A and B.

3. Construct two circles of radii 4 cm., 5 cm., such that their common chord is of length 6 cm. Measure the distance between their centres.

4. Draw $\triangle ABC$ so that $AB = AC = 5$ cm., $\angle BAC = 36°$. Construct a point D on AC such that $BD = BC$. Construct the circle ABD and measure its radius. Does it touch BC?

5. Given two points A, B and a line CD, construct a circle to pass through A and B and have its centre on CD.

[6] Draw two lines AOB, COD so that $\angle AOC = 80°$, $AO = 3$ cm., $OB = 4$ cm., $CO = 5$ cm., $OD = 2{\cdot}4$ cm. Construct a circle to pass through A, B, C. Does it pass through D?

[7] Given a circle and two points A, B inside it, construct a circle through A and B, with its centre on the given circle.

8. Draw a circle of radius 4 cm. and take a point 6 cm. from the centre. Construct the tangents from this point to the circle and measure their lengths.

[9] Draw a circle of radius 3 cm. and construct two tangents which include an angle of 75°.

10. Draw a circle of radius 3 cm. Construct a parallelogram circumscribing the circle and having one angle equal to 110°.

11. Draw a circle of radius 1·5 in. and construct a chord of the circle of length 2·5 in. Take a point A 1 in. from the centre; construct a chord of length 2·5 in. passing through A.

12. A, B, C are given points on a given circle. Construct a chord of the circle equal to AB and parallel to the tangent at C.

[13] Draw a circle of radius 3 cm. and take a point 5 cm. from the centre. Construct a chord of length 4 cm. which, when produced, passes through this point.

14. Draw a line AB of length 5 cm. and describe a circle with AB as diameter. Construct a point on AB produced such that the tangent from it to the circle is of length 3 cm.

[15] Draw a circle, centre O, radius 4 cm.; take a point A, 6 cm. from O; draw AB perpendicular to AO. Construct a point P on AB such that the tangent from P to the circle is of length 5·5 cm. Measure AP.

[16] Given a circle and a straight line, construct a point on the line such that the tangents from it to the circle contain an angle equal to a given angle. When is this impossible?

**17.** Given two parallel lines AB, CD and two points E, F, neither of which is between the two lines, construct a line FPQ cutting AB, CD at P, Q such that EP = EQ.

**18.** Draw a circle of radius 3 cm. Construct a parallelogram circumscribing the circle and such that one of its sides is 7 cm. long.

**19.** Draw two circles of radii 2 cm., 3 cm., with their centres 6·5 cm. apart. Construct their four common tangents.

[20] Draw two circles of radii 2·5 cm., 3·5 cm., touching each other externally. Construct their exterior common tangents.

[21] Draw two circles of radii 2 cm., 3 cm., with their centres 6 cm. apart. Construct a chord of the larger circle of length 4 cm. which when produced touches the smaller circle.

**22.** Draw any circle and two points A, B on it and a point C outside the circle. Construct a point P on the circle such that PC bisects $\angle$ APB.

**23.** Given a point A between two given lines BC, DE, construct points P, Q on BC, DE respectively, such that $\triangle$ APQ is equilateral. [If the circle APQ cuts BC again at R, what do you know about the angles RA, RQ make with BC?]

***24.** Draw two unequal circles intersecting at A, B. Construct, when possible, a line through A, cutting the circles at P, Q such that PQ is of given length.

***25.** Draw a circle of radius 3 cm. and construct points A, B, C on the circumference such that BC = 5 cm., BA + AC = 8·1 cm. Measure BA and AC. [If BA is produced to P so that AP = AC, $\angle$ BPC = $\frac{1}{2}\angle$ BAC.]

***26.** Construct a triangle ABC, given its perimeter, the angle BAC, and the length of the altitude AD. [See No. 2, p. 399.]

***27.** Draw two lines which meet at a point off your paper. Construct the bisector of the angle between them.

***28.** Circumscribe a square about a given quadrilateral.

***29.** Draw any circle and take two points A, B on it. Construct a point P on the circle such that chord PA equals twice chord PB.

## AREAS OF RECTANGLES

Many of the theorems relating to areas of rectangles are proved most easily by using algebraic (or trigonometric) methods. Conversely some algebraic identities may be illustrated geometrically by reference to areas of rectangles.

**Definitions.** If AB and CD are two given straight lines, any rectangle having two adjacent sides equal to AB, CD respectively is called a **rectangle contained by AB and CD:** all such rectangles are congruent.

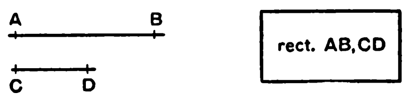



FIG. 733

A rectangle contained by AB and CD is described as the **rect.** AB, CD or as AB . CD, because the area of the rectangle is measured by the product of the measures of two adjacent sides.

The rectangle contained by AB and CD is said to be **equal** to the rectangle contained by PQ and RS **if their areas are equal,**—that is, if AB . CD = PQ . RS.

### Geometrical Illustrations of Algebraic Identities

**Examples for Oral Discussion.**

**1. Illustrate the identity**

$$k(a+b+c+d) = ka+kb+kc+kd.$$

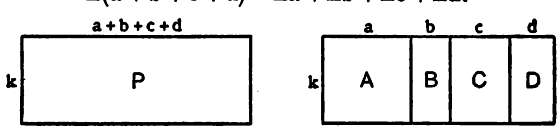



FIG. 734

Draw the rectangles shown in fig. 734.

What is the area of rectangle P?

What are the areas of the rectangles A, B, C, D?

The corresponding geometrical theorem may be stated as follows:—

**If two straight lines are given, one of which is divided into any number of parts, then the rectangle contained by the two straight lines is equal to the sum of the rectangles contained by the undivided line and each part of the divided line.**

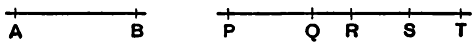



FIG. 735

In fig. 735, if AB and PQRST are straight lines,

$$AB \cdot PT = AB \cdot PQ + AB \cdot QR + AB \cdot RS + AB \cdot ST.$$

**2. Illustrate the identity**

$$(a+b)^2 = a^2 + 2ab + b^2.$$

Draw the squares in fig. 736 and divide one of them into rectangular compartments as shown.

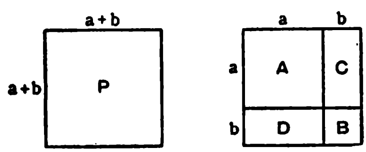



FIG. 736

What is the area of P?

What are the areas of A, B, C, D?

The corresponding geometrical theorem may be stated as follows:—

**If a straight line is divided into any two parts, the square on the whole line is equal to the sum of the squares on the two parts, together with twice the rectangle contained by these parts.**

In fig. 737, if AQB is a straight line,

$$AB^2 = AQ^2 + QB^2 + 2AQ \cdot QB.$$


FIG. 737

**3. Illustrate the identity**

$$(a - b)^2 = a^2 - 2ab + b^2.$$

Draw the squares A and B shown in fig. 738 (i).

Explain why the areas of fig. 738 (i), and (ii) are equal.

What are the areas of C, P, Q?

Hence

$a^2 + b^2 = (a - b)^2 + 2ab.$

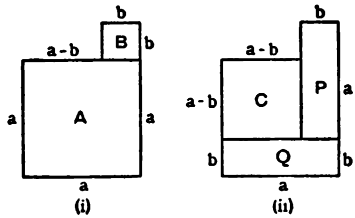



FIG. 738

The corresponding geometrical theorem may be stated as follows:

**If a straight line is divided into any two parts, the sum of the squares on the whole line and on one of the parts is equal to twice the rectangle contained by the whole line and that part together with the square on the other part.**

In fig. 737, if AQB is a straight line,

$AB^2 + QB^2 = 2AB \cdot QB + AQ^2.$


FIG. 737

**4. Illustrate the identity**

$$a^2 - b^2 = (a + b)(a - b).$$

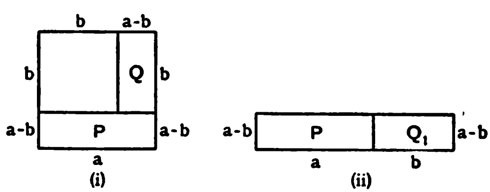



FIG. 739

Draw the square, side *a* units, shown in fig. 739 (i) and divide it into a square side *b* units and the two rectangles P and Q.

The rectangle $Q_1$ in fig. 739 (ii), is congruent to the rectangle Q in fig. 739 (i).

In fig. 739 (i), P + Q contains $(a^2 - b^2)$ units of area.

What is the area of $P + Q_1$ in fig. 739 (ii)?

The corresponding geometrical theorem may be stated as follows:—

**The difference of the squares on two straight lines is equal to the rectangle contained by the sum and the difference of the two straight lines.**

If LM and XY are two straight lines, of which LM is the greater,

$$LM^2 - XY^2 = (LM + XY)(LM - XY).$$

## Applications of Algebra to Geometry

**Example 1.** If a straight line BC is bisected at D, and if A is any other point on BC, or BC produced, then

$$AB^2 + AC^2 = 2AD^2 + 2DB^2.$$

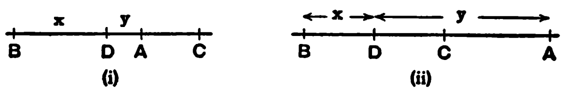



FIG. 740

Let $\quad BD = x$ in., $\quad DA = y$ in.,

$\therefore \quad DC = BD = x$ in.

Also $\quad BA = x + y$ in.

and $\quad AC = DC - DA = (x - y)$ in., fig. 740 (i),

$\quad AC = DA - DC = (y - x)$ in., fig. 740 (ii).

Hence in each case,

$$\begin{aligned} AB^2 + AC^2 &= (x^2 + y^2 + 2xy) + (x^2 + y^2 - 2xy) \text{ sq. in.} \\ &= (2x^2 + 2y^2) \text{ sq. in.} \\ &= 2DB^2 + 2AD^2. \end{aligned}$$

NOTE. Example 1 is a special case of Theorem 66, p. 419.

**Example 2.** If A, B, C, D are four points in order on a straight line, prove that

$$AC \,.\, BD = AB \,.\, CD + AD \,.\, BC.$$

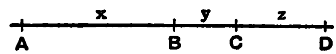



FIG. 741

Let $AB = x$ in., $BC = y$ in., $CD = z$ in.

Then $AC \,.\, BD = (x+y)(y+z)$ sq. in.

$= xy + xz + y^2 + yz$ sq. in.

and $AB \,.\, CD + AD \,.\, BC = xz + (x+y+z)y$ sq. in.

$= xz + xy + y^2 + yz$ sq. in.

$\therefore AC \,.\, BD = AB \,.\, CD + AD \,.\, BC.$

## EXERCISE 75

Illustrate by a figure the following identities:

1. $(a+b)(c+d) = ac + ad + bc + bd.$ [2] $k(a-b) = ka - kb.$

3. $(2a)^2 = 4a^2.$ [4] $(a+b+c)^2 = a^2 + b^2 + c^2 + 2bc + 2ca + 2ab.$

5. A straight line AB is bisected at O; P is any point on AO. Prove that $PO = \frac{1}{2}(PB - AP)$.

[6] A straight line AB is bisected at O and produced to P. Prove that $OP = \frac{1}{2}(AP + BP)$.

[7] ABCD is a straight line; X, Y are the mid-points of AB, CD. Prove that $AD + BC = 2XY$.

8. AD is trisected at B, C. Prove that $AD^2 = AB^2 + 2BD^2$.

9. AB is bisected at O and produced to P. Prove that

$$AO \,.\, AP = OB \,.\, BP + 2AO^2.$$

[10] AB is bisected at C and produced to Q. Prove that

$$AQ^2 = 4AC \,.\, CQ + BQ^2.$$

11. ABCD is a straight line. If $AB = CD$, prove that

$$AD^2 + BC^2 = 2AB^2 + 2BD^2.$$

12. X is a point on AB such that $AB \,.\, BX = AX^2$. Prove that

$$AB^2 + BX^2 = 3AX^2.$$

[13] C is a point on AB such that $AB \cdot BC = AC^2$. Prove that
$$AC \cdot BC = AC^2 - BC^2.$$

14. X is a point on AB such that $AB \cdot BX = AX^2$; O is the mid-point of AX. Prove that $OB^2 = 5OA^2$.

[15] AB is bisected at O and produced to P so that $OB \cdot OP = BP^2$ Prove that $PA^2 = 5PB^2$.

16. AB is produced to P so that $PA^2 = 4PB^2 + AB^2$. Prove that
$$2PA = 5PB.$$

## Projections

**Definition.** If AB and CD are any two straight lines, and if AH, BK are the perpendiculars from A, B to CD, produced if necessary, then HK is called the **projection** of AB on CD.

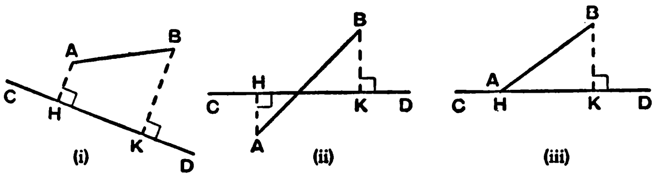



FIG. 742

In the special case shown in fig. 742 (iii), A coincides with H, and the projection of AB on CD is AK.

### Examples for Oral Discussion

1. In fig. 530, p. 270, which represents the squares on the sides of the right-angled triangle ABC, what lines represent the projection of (i) BA on QP, (ii) AC on BC, (iii) the line joining A to Q on BC?

2. In fig. 529, p. 270, which represents the squares on the sides of an acute-angled triangle ABC, what lines represent the projections of (i) BC on AB, (ii) AB on NM, (iii) AC on KH?

3. In fig. 529, p. 270, what lines represent the projections of (i) AC on EC, (ii) the line joining B to N on AC, (iii) the line joining C to K on CD?

## Co-ordinates

An important example of the use of projections is the method of fixing the position of a point in a plane by co-ordinates.

If O$x$, O$y$ are directed axes of reference, O$y$ making an angle $+90°$ with O$x$, and if P is any point in the plane of O$x$ and O$y$,

**the $x$ co-ordinate** of P is the projection OM of OP on O$x$,

and

**the $y$ co-ordinate** of P is the projection ON of OP on O$y$.

Suppose that OP is of *unit* length and makes an acute angle $\theta$ with O$x$.

Then by the definition of the sine and cosine of an *acute* angle on p. 79,

$$OM = \cos\theta \quad \text{and} \quad ON = \sin\theta.$$

$\therefore$ P is the point $(\cos\theta, \sin\theta)$.

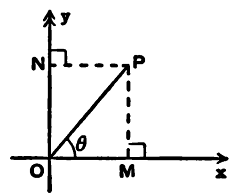


FIG. 743

Next suppose that OP is of *unit* length and makes with O$x$ an angle $\theta$ of *any magnitude*, then we make the following *definitions*:

**the x co-ordinate of P is cos $\theta$,**

and **the y co-ordinate of P is sin $\theta$.**

In fig. 744, $\theta$ is obtuse, and it is then evident that in this case the $x$ co-ordinate of P is negative and the $y$ co-ordinate of P is positive.

Hence, since $\angle MOP = 180° - \theta$,

$$\cos\theta = -\cos(180° - \theta)$$

and $$\sin\theta = \sin(180° - \theta).$$

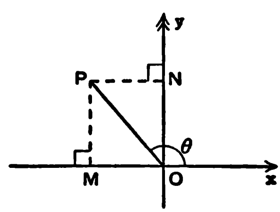


FIG. 744

These results agree with the statements on pp. 246, 271. The sines and cosines of reflex angles can be discussed in a similar way but are not needed here.

### Examples for Oral Discussion

**1. With the notation of fig. 745, where $\angle$ BAC is ACUTE, prove that**

$$a^2 = b^2 + c^2 - 2cx.$$

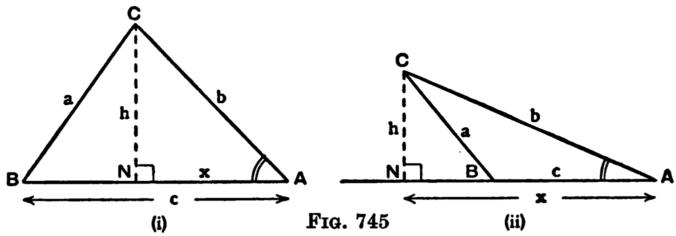



FIG. 745

In fig. 745 (i), BN $= c - x$; in fig. 745 (ii), BN $= x - c$.
In each case, apply Pythagoras to $\triangle$ BNC and $\triangle$ ANC.

**2.** Use the fact that AN is the projection of AC on AB in fig. 745 to express the result of No. 1 in words.

**3.** Prove that the result of No. 1 may be written

$$a^2 = b^2 + c^2 - 2bc \cos A.$$

**4. With the notation of fig. 746, where $\angle$ BAC is OBTUSE, prove that $a^2 = b^2 + c^2 + 2cy$.**

Apply Pythagoras to $\triangle$ BNC and $\triangle$ ANC.

**5.** Express the result of No. 4 in words.

**6.** Prove that the result of No. 4 may be written

$$a^2 = b^2 + c^2 - 2bc \cos A.$$

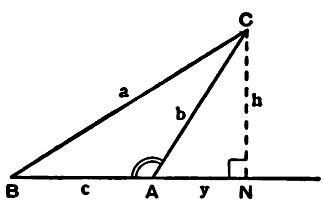



FIG. 746

Since $\angle$ CAN $= 180° - A$, $y = b \cos(180° - A) = -b \cos A$.

**The Cosine Formula.** The results established in Examples 3, 6 may be combined into the following single statement:

**In ANY triangle ABC, if BC $=$ a units, CA $=$ b units, AB $=$ c units, $a^2 = b^2 + c^2 - 2bc \cos A$.**

**Similarly,**

$$b^2 = c^2 + a^2 - 2ca \cos B \quad \text{and} \quad c^2 = a^2 + b^2 - 2ab \cos C.$$

This formula may be used to calculate

(i) the length of the third side of a triangle if the lengths of two sides and the size of the included angle are given;

(ii) any angle of a triangle if the lengths of the three sides are given.

Use the cosine formula for Nos. 7–10:

7. (i) $b=2$, $c=3$, $A=50°$, find $a$;
   (ii) $b=2$, $c=3$, $A=130°$, find $a$.

8. (i) $a=5$, $b=7$, $C=35°$, find $c$;
   (ii) $a=5$, $b=7$, $C=145°$, find $c$.

9. $a=6$, $b=5$, $c=4$, find A and C.

10. $a=7$, $b=5$, $c=4$, find A and C.

**11. If D is the mid-point of the side BC of $\triangle ABC$, prove that**

$$AB^2+AC^2=2AD^2+2BD^2.$$

Draw AN perpendicular to BC.

Complete the following:—

From $\triangle ADB$ where $\angle ADB$ is obtuse,

$$AB^2= \ . \ . \ .$$

From $\triangle ADC$ where $\angle ADC$ is acute,

$$AC^2= \ . \ . \ .$$

$$\therefore \ AB^2+AC^2= \ . \ . \ .$$

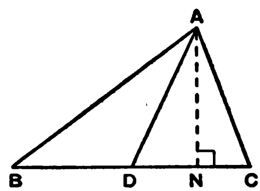


Fig. 747

12. Prove the statement in No. 11 by applying the cosine formula to $\triangle ADB$ and $\triangle ADC$.

Let $\angle ADC=\theta$; $\therefore \ \angle ADB=180°-\theta$.

**Illustrative Example.** If in $\triangle ABC$, $BC=11$ in., $CA=9$ in., $AB=6$ in., find whether $\angle BAC$ is acute or obtuse.

$$BC^2=11^2 \text{ sq. in.}=121 \text{ sq. in.},$$
$$AB^2+AC^2=(36+81) \text{ sq. in.}=117 \text{ sq. in.};$$
$$\therefore \ BC^2 > AB^2+AC^2,$$
$$\therefore \ \angle BAC \text{ is obtuse.}$$

## NUMERICAL EXAMPLES

### EXERCISE 76

[*Give answers, which are approximate, correct to three figures.*]

Find whether the triangles, the lengths of whose sides are given in Nos. 1–4, are obtuse angled or acute angled:

1. 4 in., 5 in., 7 in. 2. 8 cm., 9 cm., 12 cm.

[3] 7 in., 8 in., 11 in. [4] 15 cm., 16 cm., 22 cm.

5. ABC is an acute-angled triangle in which AB = 12 in., AC = 15 in., and the length of BC is a whole number of inches. Find the greatest possible length and the least possible length of BC.

[6] In $\triangle$ ABC, AB = 9 in., AC = 11 in., $\angle$BAC > 90°. Prove that BC > 14 in.

In Nos. 7–10, CN is an altitude of $\triangle$ ABC. Find the lengths of AN and CN and the area of $\triangle$ ABC.

7. BC = 8 in., CA = 9 in., AB = 10 in.

8. BC = 6 cm., CA = 3 cm., AB = 4 cm.

[9] BC = 7 in., CA = 13 in., AB = 10 in.

[10] BC = 11 cm., CA = 9 cm., AB = 10 cm.

[11] If AB = 6 cm., BC = 5 cm., CA = 7 cm., find the length of the projection of AB on CB.

12. If AB = 13 in., BC = 24 in., CA = 19 in., prove that the foot of the perpendicular from A to BC is a point of trisection of BC.

[13] ABCD is a parallelogram; AB = 5 in., AD = 3 in., and the projection of AC on AB is 6 in. Find AC.

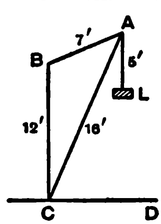



FIG. 748

14. Fig. 748 represents a crane supporting a load L; BC is vertical. Find the height of L above the horizontal plane through C.

15. The sides of a triangle are 9 cm., 7 cm., 14 cm.; find the length of the shortest median.

16. The sides of a triangle are 8 in., 9 in., 11 in.; find the length of the two shorter medians.

[17] Find the lengths of the three medians of a triangle whose sides are 6 cm., 8 cm., 9 cm.

18. AD is a median of $\triangle$ABC. If AB = 6 in., AC = 8 in., AD = 5 in., find the length of BC.

[19] In $\triangle$ABC, AB = 4 cm., BC = 5 cm., CA = 8 cm.; BC is produced to D so that BC = CD, find the length of AD.

20. The sides of a parallelogram are 5 cm., 7 cm., and one diagonal is 8 cm.; find the length of the other diagonal.

21. In $\triangle$ABC, BC = 24 cm., CA = 13 cm., AB = 17 cm.; BC is trisected at Y, Z. Find the lengths of AY, AZ.

22. In $\triangle$ABC, AC = 8 cm., BC = 6 cm., $\angle$ACB = 120°. Find the length of AB.

23. In $\triangle$ABC, AB = 8 cm., AC = 7 cm., BC = 3 cm. Prove that $\angle$ABC = 60°.

[24] In $\triangle$ABC, AB = 14 in., BC = 10 in., CA = 6 in. Prove that $\angle$ACB = 120°.

*25. In $\triangle$ABC, BC = $(2a + 4b)$ in., CA = $(4a + b)$ in., AB = $(2a + 3b)$ in., find in terms of $a$, $b$, the length of the median AD.

*26. AD is a median of $\triangle$ABC, and DN is the perpendicular from D to AB. If AB = 12 in., AC = 8 in., AD = 6 in., find the length of AN.

*27. In $\triangle$ABC, AB = AC = 3 cm., BC = 2 cm.; D is taken on BC produced so that AD = $6\sqrt{2}$ cm. Prove that $\angle$BAD is a right angle.

*28. In $\triangle$ABC, $AB^2 - AC^2 = 66$ sq. in. and BC = 6 in. Find the length of the projection of AC on BC.

**Historical Note.** Theorem 66, by means of which it is possible to calculate the lengths of the medians of a triangle, whose sides are given, is associated with the name of Apollonius of Perga (247–205 B.C.), known among the ancients as the "Great Geometer." His writings, which dealt mainly with the properties of the ellipse, parabola, and hyperbola, see p. 216, and those of Euclid, dominated geometry up to the beginning of the nineteenth century.

## THEOREM 64

**In an obtuse-angled triangle, the square on the side opposite the OBTUSE angle is equal to the sum of the squares on the sides containing it PLUS twice the rectangle contained by one of those sides and the projection on it of the other.**

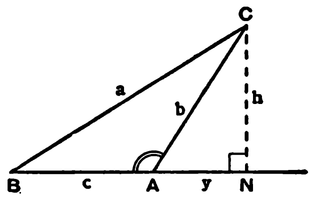



FIG. 749

**Given** a triangle ABC in which $\angle$BAC is obtuse, and the perpendicular CN from C to BA produced.

**To prove that** $BC^2 = BA^2 + CA^2 + 2BA . AN$.

**Proof.** (Put in a small letter for each length that comes in the answer and also for the altitude.)

Let BC $= a$ units, CA $= b$ units, AB $= c$ units, AN $= y$ units, CN $= h$ units.

Since $\angle$BNC is a right angle,

$$a^2 = (c+y)^2 + h^2 \qquad \textit{Pythagoras},$$

$$\therefore\ a^2 = c^2 + 2cy + y^2 + h^2.$$

Since $\angle$ANC is a right angle,

$$b^2 = y^2 + h^2 \qquad \textit{Pythagoras},$$

$$\therefore\ a^2 = c^2 + 2cy + b^2,$$

that is, $BC^2 = BA^2 + 2BA . AN + CA^2$.

NOTE. Some pupils find it difficult to remember this result; it is a useful mnemonic to observe that the vertex A, opposite the side BC, is one extremity of each length that occurs on the right side.

## THEOREM 65

**In any triangle, the square on the side opposite an ACUTE angle is equal to the sum of the squares on the sides containing it MINUS twice the rectangle contained by one of those sides and the projection on it of the other.**

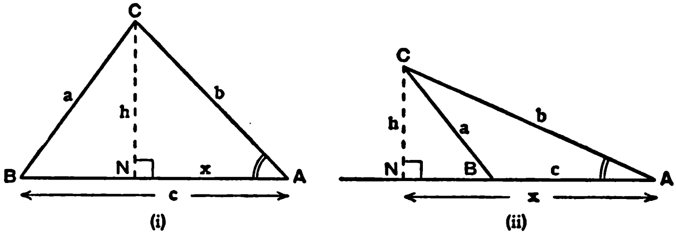



FIG. 750

**Given** a triangle ABC in which $\angle$BAC is acute, and the perpendicular CN from C to AB or AB produced.

**To prove that** $BC^2 = BA^2 + CA^2 - 2BA \cdot AN$.

**Proof.** (Put in a small letter for each length that comes in the answer and also for the altitude.)

Let BC $= a$ units, CA $= b$ units, AB $= c$ units, AN $= x$ units, CN $= h$ units.

In fig. 750 (i), BN $= c - x$ units;
in fig. 750 (ii), BN $= x - c$ units.

Since $\angle$CNB is a right angle,

$a^2 = (c-x)^2 + h^2$ in fig. 750 (i) *Pythagoras,*
$a^2 = (x-c)^2 + h^2$ in fig. 750 (ii) *Pythagoras,*

$\therefore$ in each case, $a^2 = c^2 - 2cx + x^2 + h^2$.

Since $\angle$CNA is a right angle,

$b^2 = x^2 + h^2$ *Pythagoras,*

$\therefore$ $a^2 = c^2 - 2cx + b^2$,

that is, $BC^2 = BA^2 - 2BA \cdot AN + CA^2$.

NOTE. As for Theorem 64, it is a useful mnemonic to observe that the vertex A, opposite to the side BC, is one extremity of each length that occurs on the right side.

## THEOREM 66 (Apollonius' Theorem)

**In any triangle, the sum of the squares on two sides is equal to twice the square on half the third side together with twice the square on the median which bisects the third side.**

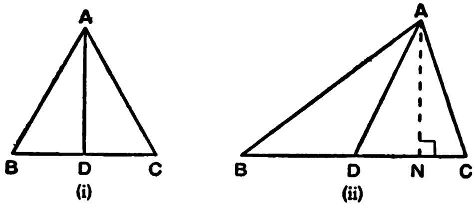



FIG. 751

**Given** a triangle ABC in which D is the mid-point of BC.

**To prove that** $AB^2 + AC^2 = 2AD^2 + 2BD^2$.

**Proof.** Either (i) $\angle$s ADB, ADC are both right angles
or (ii) one is obtuse and the other acute.

(i) If $\angle$ADB and $\angle$ADC are right angles,

$AB^2 = AD^2 + BD^2$ and $AC^2 = AD^2 + DC^2$ *Pythagoras.*

But $BD = DC$ *given,* $\therefore$ $BD^2 = DC^2$.

$\therefore$ by addition, $AB^2 + AC^2 = 2AD^2 + 2BD^2$.

(ii) Suppose $\angle$ADB is obtuse, then $\angle$ADC is acute.
Let AN be the perpendicular from A to BC.
From $\triangle$ADB, since $\angle$ADB is obtuse,

$$AB^2 = AD^2 + BD^2 + 2BD \cdot DN.$$

From $\triangle$ADC, since $\angle$ADC is acute,

$$AC^2 = AD^2 + DC^2 - 2DC \cdot DN.$$

But $BD = DC$ *given,*

$\therefore$ $BD^2 = DC^2$ and $BD \cdot DN = DC \cdot DN$.

$\therefore$ by addition, $AB^2 + AC^2 = 2AD^2 + 2BD^2$.

If $\angle$ADB is acute, then $\angle$ADC is obtuse, and the proof is the same as before, except that B and C are interchanged.

## EXERCISE 77

1. $BE$, $CF$ are altitudes of the acute-angled triangle $ABC$. Write down two expressions for $BC^2$ not involving $BE$ or $CF$. Hence prove that $AF \cdot AB = AE \cdot AC$.

[2] If the altitudes $BE$, $CF$ of the acute-angled triangle $ABC$ cut at $H$, prove that $BH \cdot HE = CH \cdot HF$. [Use $\triangle BHC$ to write down two expressions for $BC^2$.]

3. $ABC$ is an equilateral triangle; $BC$ is produced to $D$ so that $BC = CD$. Prove that $AD^2 = 3AB^2$. [Use Apollonius.]

[4] In $\triangle ABC$, $AB = AC$; $AB$ is produced to $D$ so that $AB = BD$. Prove that $CD^2 = AB^2 + 2BC^2$.

5. $ABCD$ is a parallelogram. Prove that

$$AC^2 + BD^2 = 2AB^2 + 2BC^2.$$

[Let $AC$ cut $BD$ at $K$.]

[6] $ABCD$ is a rectangle; $P$ is any point in the same plane or in any other plane. Prove that $PA^2 + PC^2 = PB^2 + PD^2$. [Let $AC$ cut $BD$ at $K$; join $PK$.]

7. $A$, $B$ are fixed points; $P$ is a variable point such that $PA^2 + PB^2$ is constant. Find the locus of $P$.
[Join $P$ to the mid-point of $AB$.]

8. $AX$, $BY$ are medians of $\triangle ABC$. Prove that

$$AX^2 - BY^2 = \tfrac{3}{4}(AC^2 - BC^2).$$

9. In $\triangle ABC$, $AB = AC$; $CD$ is an altitude Prove that

$$BC^2 = 2AB \cdot BD.$$

[10] In $\triangle ABC$, $\angle A = 90°$; $AC$ is produced to $D$ so that $CD = BC$. Prove that $BD^2 = 2BC \cdot AD$.

[11] $A$, $B$ are fixed points; $P$ is a variable point such that $PA^2 + PB^2$ is constant. Prove that the area of $\triangle PAB$ is greatest when $PA = PB$.

12. (i) If $AD$ is a median of $\triangle ABC$ and if the lengths of the sides $BC$, $CA$, $AB$ are $a$, $b$, $c$ respectively, find $AD^2$ in terms of $a$, $b$, $c$.
(ii) If $AD$, $BE$, $CF$ are the medians of $\triangle ABC$, prove that

$$4(AD^2 + BE^2 + CF^2) = 3(BC^2 + CA^2 + AB^2).$$

13. The base $BC$ of $\triangle ABC$ is trisected at $X$ and $Y$. Prove that $AX^2 + AY^2 + 4XY^2 = AB^2 + AC^2$.

[14] In $\triangle ABC$, $\angle C = 90°$; $AB$ is trisected at $P$ and $Q$. Prove that $PC^2 + CQ^2 + QP^2 = \tfrac{2}{3}AB^2$.

15. BC is a diameter of a circle, centre O; A is any point inside the circle on the radius OE. Prove that

$$2AE^2 = AB^2 + AC^2 - 2AO \cdot BC.$$

[16] In $\triangle ABC$, $AB = AC$. From any point D on AB a line DE is drawn parallel to BC to cut AC at E. Prove that

$BE^2 = CE^2 + BC \cdot DE$. [Draw DH, EK perpendicular to BC.]

*17. BE, CF are altitudes of the acute-angled triangle ABC. Prove that $BA \cdot BF + CA \cdot CE = BC^2$.

*18. ABC is a triangle; ABPQ, ACXY are squares outside $\triangle ABC$. Prove that $BC^2 + QY^2 = AP^2 + AX^2$.

*19. ABCD is a quadrilateral; X, Y are the mid-points of AC, BD. Prove $AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2 + 4XY^2$.

*20. ABCD is a tetrahedron; $\angle BAC = \angle CAD = \angle DAB = 90°$. Prove that BCD is an acute-angled triangle.

*21. D is a point on the side BC of $\triangle ABC$ such that $BD = 2DC$. Prove that $AB^2 + 2AC^2 = 6CD^2 + 3AD^2$.

## Segments of a Straight Line

If AB is any straight line and if P is any point on AB between the points A and B, AB is said to be **divided internally** at P, see fig. 752 (i).

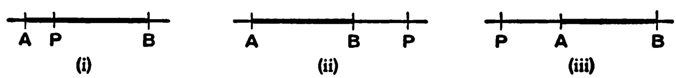



FIG. 752

If P is any point beyond B on AB produced, fig. 752 (ii), or beyond A on BA produced, fig. 752 (iii), AB is said to be **divided externally** at P.

In each case, that is whether P lies on AB or on AB produced or on BA produced, PA and PB are called **segments of the straight line** AB; and a rectangle, whose adjacent sides are equal to PA and PB, is said to be **contained by the segments** PA, PB **of the straight line divided at** P.

NOTE. Each segment of the line AB, divided at P, is measured from P.

## Intersecting Chords of a Circle

**Examples for Oral Discussion.**

1. Through any point X inside a circle, centre O, radius r, a line is drawn cutting the circle at A and B. Prove that

$$XA \cdot XB = r^2 - OX^2.$$

With the notation in fig. 753, express the lengths of XA, XB in terms of $a$, $x$. Use Pythagoras for $\triangle$ONA, $\triangle$ONX.

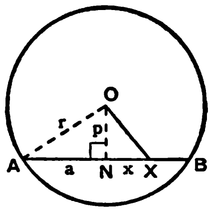



FIG. 753

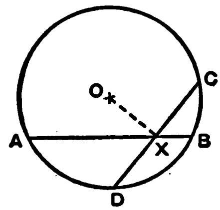



FIG. 754

2. Two chords AB, CD of a circle intersect at a point X inside the circle. Prove that XA . XB = XC . XD.

Express this fact in words.

Note that the segments of the chord AB, divided at X, are the lines XA, XB measured from X.

3. From any point X outside a circle, centre O, radius r, a line is drawn cutting the circle at A and B. Prove that

$$XA \cdot XB = OX^2 - r^2.$$

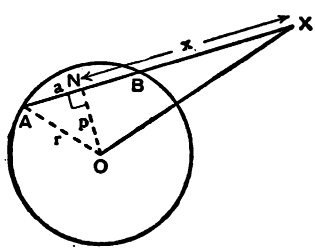



FIG. 755

Use the method given above for (1).

**4. Two chords AB, CD of a circle intersect, when produced, at a point X outside the circle; XT is the tangent from X to the circle. Prove that**

$$XA \,.\, XB = XC \,.\, XD = XT^2.$$

**Express these results in words.**

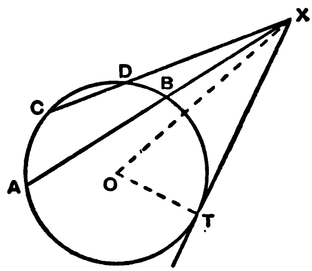



FIG. 756

Note that the segments of the chord AB, divided at X, are the lines XA, XB measured from X.

5. State the converse of No. 2 and prove that it is true.

6. State the converse of No. 4 and prove that it is true.

## NUMERICAL EXAMPLES

### EXERCISE 78

Nos. 1–5 refer to fig. 754 in which AXB, CXD are two chords of a circle, centre O.

1. If AX = 6 in., XB = 2 in., CX = 3 in., find XD.

2. If AB = 11 cm., BX = 3 cm., CX = 4 cm., find CD.

[3] If AX = 12 cm., AB = 15 cm., CX = XD, find CD.

4. If OA = 6 in., AX = 5 in., OX = 4 in., find BX.

[5] If OA = 7 in., OX = 5 in., find AX . XB.

Nos. 6–9 refer to fig. 756 in which the chords AB, CD of a circle, centre O, meet at a point X outside the circle and XT is a tangent.

6. If $CD = 2$ in., $DX = 6$ in., $BX = 3$ in., find AB.

7. If $AB = 9$ cm., $BX = 3$ cm., find TX.

[8] If $TX = 5$ in., $DX = 2\frac{1}{2}$ in., find CD.

9. If $OA = 5$ cm., $OX = 9$ cm., find $XA \cdot XB$.

10. From a point P on a circle PN is drawn perpendicular to a diameter AB; $AN = 4$ in., $NB = 16$ in., find PN.

[11] In $\triangle ABC$, $\angle BAC = 90°$, $AB = 4$ cm., $AC = 3$ cm.; AD is an altitude; find BD.

12. In $\triangle ABC$, $AB = 9$ cm., $AC = 12$ cm., F is the mid-point of AC. If the circle through B, F, C cuts AB at E, find BE.

13. AOB, COD are two straight lines such that $AB = 20$ cm., $CD = 19$ cm., $AO = 6$ cm., $CO = 7$ cm. Prove that ACBD is cyclic.

[14] OAB, OCD are two straight lines such that $OA = 3$ cm., $AB = 12$ cm., $OC = 5$ cm., $CD = 4$ cm. Prove that ABDC is cyclic.

15. In $\triangle ABC$, $AB = 4$ cm., $BC = 8$ cm.: D is a point on BC such that $DC = 6$ cm. Prove that AB touches the circle ADC

In Nos. 16–19, figs. 757–760, PT represents the tangent at T. The unit of length is 1 cm. Find the unknown lengths.

16.

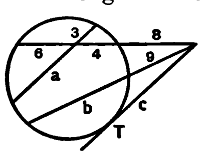



FIG. 757

[17]

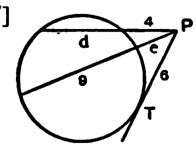



FIG. 758

[18]

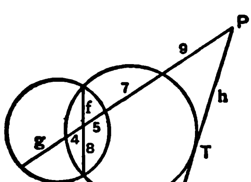



FIG. 759

19.

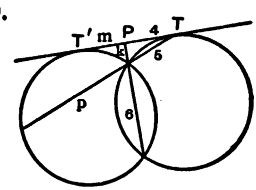



FIG. 760

20. The diagonals AC, BD of the parallelogram ABCD are of lengths 8 cm., 10 cm. The circle BCD cuts CA at F. Find AF.

21. The roadway ACB of a bridge is a circular arc resting on supports at A, B at the same level; the highest point C of the roadway is 4 ft. above AB, and AB is 8 yd. Find the radius of the arc.

22. ABC is a triangle inscribed in a circle; AB = AC = 10 cm., BC = 16 cm.; AD is drawn perpendicular to BC and is produced to meet the circle at E. Find DE and the radius of the circle.

[23] Given that the straight line which joins two points on the surface of level water is two miles long and is 8 in. below the surface of the water at its middle point, find the radius of the Earth in miles.

24. AXB, CXD are two perpendicular chords of a circle, centre O; AX = 3 in., CX = 5 in., XD = 6 in. Find OX and the radius of the circle.

[25] A small heavy body is suspended from a fixed point by a string $6\frac{1}{2}$ ft. long; it is pulled aside, the string remaining taut, so that it rises 6 in. Through what *horizontal* distance does it move?

26. In $\triangle$ABC, AB = 9 in., AC = 15 in., $\angle$A = 90°. Find the diameter of the circle which touches AC at C and passes through B.

[27] In $\triangle$ABC, AB = 6 cm., AC = 4 cm., $\angle$A = 90°. Find the radius of the circle which touches AB at B and passes through C.

*28. (i) If the mean radius of the Earth is $r$ miles and if a man stands on a hill of height $h$ miles above mean level, show that the distance he can see is about $\sqrt{(2rh)}$ miles.

(ii) Taking the radius of the Earth as 3960 miles, show that at a height of $x$ *feet* above sea-level, the distance that can be seen across level ground is about $\sqrt{(\frac{3}{2}x)}$ miles.

(iii) Find the approximate distance of the horizon for a height of (i) 6 ft., (ii) 600 ft.

## THEOREM 67

**If two chords of a circle intersect at a point inside the circle, the rectangle contained by the segments of the one is equal to the rectangle contained by the segments of the other.**

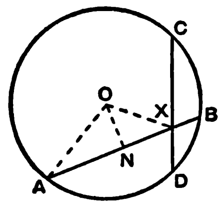



FIG. 761

**Given** two chords AB, CD of a circle, intersecting at a point X inside the circle.

**To prove that** $XA \cdot XB = XC \cdot XD$.

**Construction.** Draw the perpendicular ON from the centre O to AB.
Join OA, OX.

**Proof.** $AN = NB$ *perp. from centre bisects chord,*

$\therefore XA \cdot XB = (AN + NX)(NB - NX)$
$= (AN + NX)(AN - NX)$
$= AN^2 - NX^2$.

But $AN^2 = OA^2 - ON^2$
and $NX^2 = OX^2 - ON^2$ *Pythagoras.*

$\therefore AN^2 - NX^2 = OA^2 - OX^2$,
$\therefore XA \cdot XB = OA^2 - OX^2$.

Similarly, it may be proved that
$XC \cdot XD = OC^2 - OX^2$.

But $OA = OC$ *radii,*

$\therefore XA \cdot XB = XC \cdot XD$.

**Corollary.** If X is any point inside a circle, centre O, radius $r$, the rectangle contained by the segments of any chord drawn through X equals $r^2 - OX^2$.

## THEOREM 68

**If two chords of a circle, when produced, intersect at a point outside the circle, the rectangle contained by the segments of the one is equal to the rectangle contained by the segments of the other and each rectangle is equal to the square on the tangent from the point of intersection to the circle.**

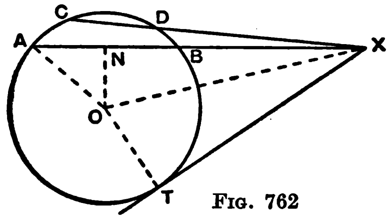



FIG. 762

**Given** two chords AB, CD of a circle, intersecting at a point X outside the circle, and the tangent XT.

**To prove that** $XA \cdot XB = XC \cdot XD = XT^2$.

**Construction.** Draw the perpendicular ON from the centre O to AB. Join OA, OX, OT.

**Proof.** $AN = NB$ *perp. from centre bisects chord,*

$\therefore XA \cdot XB = (XN + NA)(XN - NB)$

$= (XN + NA)(XN - NA) = XN^2 - NA^2.$

But $XN^2 = XO^2 - ON^2$

and $NA^2 = AO^2 - ON^2$ *Pythagoras.*

$\therefore XN^2 - NA^2 = XO^2 - AO^2,$

$\therefore XA \cdot XB = XO^2 - AO^2.$

But the tangent XT is perpendicular to the radius OT,

$\therefore XT^2 = XO^2 - TO^2$ *Pythagoras,*

$= XO^2 - AO^2$ $OT = OA$, *radii,*

$= XA \cdot XB.$

Similarly, it may be proved that $XT^2 = XC \cdot XD$.

**Corollary.** If X is any point outside a circle, centre O, radius $r$, the rectangle contained by the segments of any chord drawn through X equals $OX^2 - r^2$.

## THEOREM 69

**If two straight lines AB and CD are divided both internally or both externally at the same point X such that XA . XB = XC . XD, the four points A, B, C, D are concyclic.**

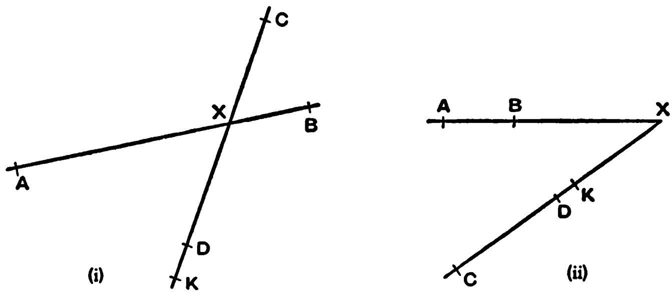



FIG. 763

**Construction and Proof.** Draw the circle ABC and let it cut the line CD, produced if necessary, at K.

If X divides AB internally, fig. 763 (i), X lies inside the circle ABC. $\therefore$ X divides the chord CK internally. If X divides AB externally, fig. 763 (ii), X lies outside the circle ABC, $\therefore$ X divides the chord CK externally. $\therefore$ in each case, K and D are on the same side of X.

Also XA . XB = XC . XK  A, C, B, K *concyclic*,
and XA . XB = XC . XD  *given*,

$\therefore$ XC . XK = XC . XD, $\therefore$ XK = XD.

But K and D are on the same side of X,
$\therefore$ K and D are the same point.
$\therefore$ the circle ABC passes through D.

**Corollary.** **If the straight line AB is divided externally at X, and if C is a point, not on AB, such that XA . XB = $XC^2$, the circle ABC touches XC at C.**

The proof is the same as for fig. 763 (ii).

There are two other important results which it is convenient to refer to at this stage: they are given later as corollaries to Theorem 78, p. 504.

**If AB is a diameter of a circle and if PN is the perpendicular to AB from any point P on the circumference, then**

(1) $PN^2 = AN . NB$;

(2) $AP^2 = AN . AB$ and $BP^2 = BN . BA$.

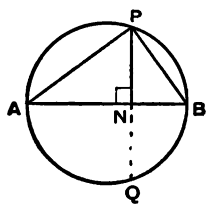


FIG. 764

(1) Produce PN to meet the circle again at Q.

Since PQ is perpendicular to the diameter AB,

$$PN = NQ;$$

$$\therefore AN . NB = PN . NQ = PN . PN$$
$$= PN^2.$$

(2) Since $\angle PNB$ is a right angle,

the circle on PB as diameter passes through N.

Since $\angle APB$ is a right angle,

AP is the tangent at P to the circle on PB as diameter.

$$\therefore AP^2 = AN . AB.$$

Similarly, it may be proved that $BP^2 = BN . BA$.

NOTE. Result (2) also follows at once from the figure and proof of Pythagoras' theorem, see p. 283, and it is a useful exercise for the reader to prove result (1) by applying Pythagoras to the three triangles APB, ANP, BNP. [Let $AN = x$, $NB = y$, $PN = h$, $PA = b$, $PB = a$.]

Both results can be proved easily by using trigonometry.

## EXERCISE 79

1. Two circles cut at A and B; P is any point on AB produced. Prove that the tangents from P to the two circles are equal.

[2] Prove that the common chord of two intersecting circles, when produced, bisects the common tangents.

3. The tangent at P to the circle APB meets AB produced at T. If $TA = 2TP$, prove that $AB = 3BT$.

[4] P and Q are points on the chords AB and AC respectively of the circle ABC, centre O, such that $AP . PB = AQ . QC$. Prove that $OP = OQ$. [Use the Corollary of Theorem 67.]

5. If the altitudes BE, CF of $\triangle ABC$ intersect at H, prove that (i) $BH . HE = CH . HF$; (ii) $AF . AB = AE . AC$.

[6] AB is a diameter of the circle APB; a line perpendicular to AB cuts AB, AP at H, K respectively. Prove that

$$AH . AB = AK . AP.$$

7. K is a point inside $\triangle ABC$; BK, CK, produced, cut AC, AB at Q, R. If $BK . KQ = CK \; KR$, prove $AR . AB = AQ . AC$.

[8] BE, CF are altitudes of $\triangle ABC$; M, N are the mid-points of AC, AB respectively. Prove that $AM \; AE = AN . AF$. What follows from this fact?

9. AB, AC are two chords of a circle; any line parallel to the tangent at A cuts AB, AC at D, E respectively. Prove that

$$AB . AD = AC . AE.$$

[10] AB is a diameter of the circle APQB; the tangent at B meets AP, AQ produced at X, Y respectively. Prove that

$$AP \; AX = AQ . AY = AB^2.$$

11. AB, AC are two equal chords of a circle; AP is another chord of the circle which cuts BC at Q. Prove $AP . AQ = AB^2$.

[12] Two lines XAB, XCD cut a circle at A, B, C, D; through X a line is drawn parallel to BC to meet DA produced at Y. Prove that $YX^2 = YA . YD$.

13. In $\triangle ABC$, $AB = AC$ and $\angle A = 36°$. If the bisector of $\angle ABC$ meets AC at P, prove that $AC . CP = BC^2 = AP^2$.

[14] In $\triangle ABC$, $\angle BAC = 90°$ and $AB = 2AC$. If $AD$ is an altitude, prove that $BD = 4DC$.

15. Any two circles being given, a third circle is drawn cutting one of the circles at A, B and the other at C, D; AB and CD, when produced, meet at X. Prove that the tangents from X to the three circles are equal.

16. ABYX and ABZ are two circles; AB and XY, when produced, meet at T; TZ is the tangent from T to the circle ABZ. Prove that the circle XYZ touches TZ.

[17] PQ is a chord of a circle, centre O; the tangents at P, Q meet at X; OX cuts PQ at N. Prove that $ON \cdot OX = OP^2$.

[18] Two circles intersect at A, B; X is a point such that the tangents from X to the circles are equal. Prove that X must lie on AB produced or BA produced.

19. In fig. 765, CE touches the circle BAED, and DF touches the circle CAB. If CAD is a straight line, prove that

$$CE^2 + DF^2 = CD^2.$$

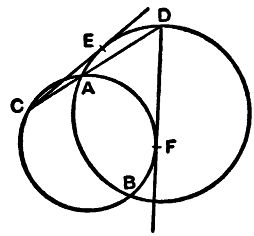



FIG. 765

20. Three circles are drawn so that each intersects the other two. Prove that the three common chords are concurrent. (If AB, CD, EF are the common chords and if AB cuts CD at X, suppose if possible EX when produced cuts the circles at distinct points P, Q.)

21. TP, TQ are tangents to a circle HPKQ, centre O; TO cuts PQ at N. If HNK is a straight line, prove that the circle THK passes through O. [Prove that O, P, T, Q are concyclic.]

[22] Two circles cut at A, B; X is any point on AB produced; a circle, centre X, cuts one circle at P, Q and the other at L, M; XP, XM, produced if necessary, cut the circles PQA, LMA at S, T. Prove that $PS = TM$.

23. AH, AK are diameters of the circles AQH, APK. If PAH, QAK are straight lines, prove that $PA \cdot AH = QA \cdot AK$.

[*This exercise is continued on p.* 433.]

## CONSTRUCTION 17

**Construct a square equal in area to a given rectangle.**

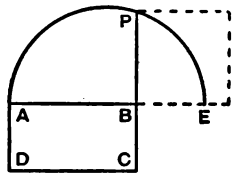



FIG. 766

**Given** a rectangle ABCD.

**To construct** a square equal in area to ABCD.

**Construction.** Produce AB to E making BE = BC.

On AE as diameter describe a semicircle.

Produce CB to meet the semicircle at P.

On BP describe a square.

This is the required square.

**Proof.** Since AE is a diameter of the circle APE and since PB is the perpendicular from P to AE,

$$BP^2 = AB \cdot BE.$$

But $\quad BE = BC \qquad$ *constr.*,

$$\therefore BP^2 = AB \cdot BC$$

$$= \text{area of } ABCD.$$

NOTE. The proof of Construction 17 depends on the property proved on p. 429.

The method is the same as that used for constructing the mean proportional between two given lines, see p. 514, first method.

Construction 17 may be used *to construct a square equal in area to any polygon.*

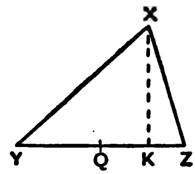


FIG. 767

**Construction.** By the method of Construction 9, p. 263, reduce the polygon to an equivalent triangle XYZ.
Draw the altitude XK and bisect YZ at Q.
Use Construction 17 to construct a square equal in area to the rectangle contained by YQ and XK.
This is the required square.

**Proof.** Area of polygon = area of $\triangle$ XYZ
$= \frac{1}{2}$YZ . XK
$=$ YQ . XK
$=$ square.

---

EXERCISE 79 (*continued*)

*24. The triangle ABC is such that AC equals the diagonal of the square described on AB; D is the mid-point of AC. Prove $\angle$ ABD $= \angle$ ACB.

*25. In $\triangle$ ABC, $\angle$ BAC $= 90°$; E is a point on BC such that AE = AB. Prove that BE . BC = 2AB$^2$. [Draw AN perpendicular to BC.]

*26. P, Q, R are points on the sides BC, CA, AB of $\triangle$ ABC such that BP . PC = CQ . QA = AR . RB. Prove that the circles ABC, PQR are concentric.

*27. AB is a diameter of the circle APQB; AP $= \frac{1}{3}$AB; N is the mid-point of PB and ANQ is a straight line. Prove that NQ $= \frac{2}{3}$AN. [Use Apollonius for $\triangle$ APB.]

## CONSTRUCTION 18

**Construct a circle to pass through two given points and to touch a given line, not parallel to the line joining the given points.** [Two solutions.]

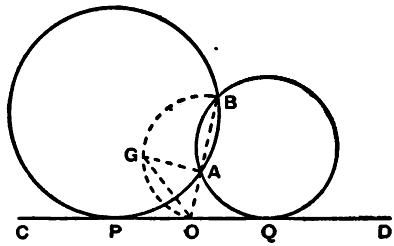



FIG. 768

**Given** two points A, B and a line CD not parallel to AB.

**To construct** a circle to pass through A, B and touch CD.

**Construction.** Join BA and produce it to meet CD at O.
Describe a semicircle with OB as diameter.
Through A draw AG perpendicular to OB to meet the semicircle at G.
Join OG. With centre O, radius OG, describe a circle to cut CD at P, Q.
Construct the circles through A, B, P and A, B, Q.
These are the required circles.

**Proof.** Since OB is a diameter of the circle OGB and since GA is the perpendicular from G to OB,

$$OA \,.\, OB = OG^2$$
$$= OP^2 \qquad \textit{constr.},$$

$\therefore$ OP is the tangent at P to the circle ABP.

Similarly, it may be proved that OQ is the tangent at Q to the circle ABQ.

NOTE. This method fails if AB is parallel to CD. This special case forms an easy exercise: only one circle can then be drawn through A and B to touch CD.

## EXERCISE 80

1. Draw a rectangle, 5 cm. by 8 cm., and construct a square of equal area. Measure its side.

[2] Construct a line of length $\sqrt{(31)}$ cm. and measure it.

3. Construct a square equal in area to an equilateral triangle of side 3 in. Measure its side.

[4] Draw a regular hexagon of side 5 cm. and construct a square of equal area. Measure its side.

5. Draw a quadrilateral ABCD in which AB = 3 cm., BC = 4 cm., CD = DA = 6·5 cm., $\angle$ABC = 90°. Construct a square equal in area to ABCD and measure its side.

6. Use construction 17 to solve the simultaneous equations,

$$x + y = 11,\ xy = 24.$$

[7] Use construction 17 to solve the equation, $x(9 - x) = 12$.

8. A, B are given points and CD is a given line parallel to AB. Show how to construct a circle to pass through A and B and to touch CD.

[9] Draw a line AB. Construct a point P on AB such that $AP^2 = \frac{2}{3}AB^2$.

10. Given a rectangle ABCD, construct when possible a point P on AB such that $AP \cdot PB = BC^2$. When is this impossible?

11. Construct a circle to pass through two given points A, B and to touch a given circle. [Draw any circle through A, B and let it cut the given circle at P, Q; let AB and PQ meet at O. From O draw OH, OK to touch the given circle, and prove that $OH^2 = OA \cdot OB$.]

12. Given four points A, B, C, D on a straight line, construct when possible a point P on the line such that $PA \cdot PD = PB \cdot PC$.

*13. Given a rectangle ABCD, construct a point P on AB produced such that $PA \cdot PB = BC^2$. [Draw a tangent QT to the circle AQB, diameter AB, making QT = BC; find P on AB so that the tangent PK equals QT. *Or* bisect AB at N and use the fact that $PA \cdot PB = PN^2 - AN^2$.]

*14. Show how to construct a circle to pass through two given points and to cut a given circle so that the common chord is of given length.

*15. Construct a circle to pass through a given point A and to touch two given lines BC, BD. [Draw the bisector BE of $\angle$CBD and take the image A′ of A in BE.]

# REVISION OF GEOMETRICAL FACTS FOR ORAL DISCUSSION

## EXERCISE 81

Nos. 1–10 refer to fig. 769 which represents a quadrilateral and its diagonals.

1. (i) What is the definition of a parallelogram?
   (ii) If ABCD is a parallelogram, what facts do you know about lengths and about areas?
   (iii) State as many tests as you can for ABCD to be a parallelogram.

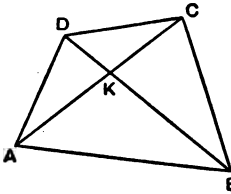


Fig. 769

2. What facts are true (i) for a rectangle, (ii) for a rhombus, which are not true for all parallelograms?

3. What triangles are equal in area if AD is parallel to BC?

4. What triangles are equal in area if BK = KD?

5. What angles are equal if a circle can be drawn through A, B, C, D?

6. What can you say about the sizes of angles if A lies outside the circle on BD as diameter and if C lies on it?

7. What angles are equal if DA touches the circle DKC?

8. If AK = KC, what do you know about $DA^2 + DC^2$?

9. What follows if AB is greater than BC?

10. What follows if BA = BC and DA = DC?

Nos. 11–14 refer to fig. 770.

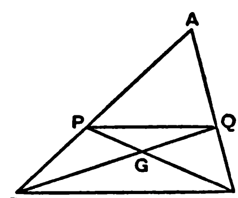


Fig. 770

11. State all the facts you know relating to lengths and areas if AP = PB and AQ = QC.

12. If AP = PB and AQ = QC and if AG produced cuts BC at X, what facts about lengths do you know?

13. What can you say about the figure if the areas of △ APC, △ AQB are equal?

14. State all the facts you know relating to lengths and areas if AP = 2PB and if PQ is parallel to BC.

Draw figures to show what is meant by the phrases in Nos. 15–30 and state what facts you associate with them.

**15.** Alternate angles.

**16.** Exterior angle of triangle.

**17.** The ambiguous case.

**18.** Mid-point theorem.

**19.** Intercept theorem.

**20.** Altitude of triangle.

**21.** The distance of a point from a straight line.

**22.** Triangles and parallelograms on the same base and between the same parallels.

**23.** Reduction of a quadrilateral to an equivalent triangle.

**24.** The angle at the centre of a circle.

**25.** Angles in the same segment.

**26.** A cyclic quadrilateral.

**27.** Alternate segment theorem.

**28.** (i) Circumcentre, (ii) In-centre, (iii) Orthocentre, (iv) Centroid, of any triangle.

**29.** Extensions of Pythagoras' theorem.

**30.** Segments of chords of a circle.

**31.** Make a list of all the standard loci you know associated with either lengths or angles or areas. Illustrate them by figures.

# REVISION OF RIDER WORK FOR ORAL DISCUSSION

## EXERCISE 82

Nos. 1–12 refer to fig. 771 which represents a triangle and three concurrent lines through its vertices.

**1.** If $y = z$, prove that $m = n$.

**2.** What angles are equal if $y + z = 2$ right angles?

**3.** If $\angle PDC = \angle DBC$, find an angle equal to $\angle DPC$.

**4.** Express $\angle BDC - \angle BAC$ as the sum of two angles.

**5.** If $DB = DC$ and $DR = DQ$, prove that $AB = AC$.

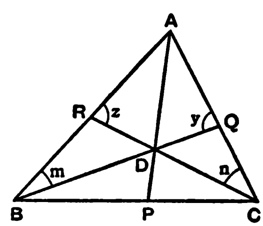


Fig. 771

6. Is it possible to draw the figure so that BD = DQ and also CD = DR?

7. Is it possible to draw the figure so that BP = PC and also BD = DQ?

8. If RD = RB and BD = DC, compare the sizes of ∠ RBC and ∠ RCB.

9. If BD = DC = CP, find the relation between ∠ BDP and ∠ CDP.

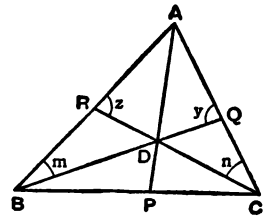


FIG. 771

10. How must the figure be drawn if
  (i) △ BPA, △ BQA are equal in area?
  (ii) △ PDC, △ RDA are equal in area?
  (iii) △ BDC, △ BDA are equal in area?

11. If BD and AD are the bisectors of ∠ CBA, ∠ CAB, find the relation between ∠ BDA and ∠ BCA.

12. If AR = AD, express ∠ BCR in terms of ∠ ABP and ∠ APB.

Nos. 13–17 refer to fig. 772 in which ABCD is a parallelogram and Y is the mid-point of CD.

13. Prove that BZ = 2AD.

14. Prove that DZ is parallel to AC.

15. Prove that
  △ ABZ = quad. ABCD.

16. Prove that
  △ ABD = ½ quad. ACZD.

17. Join KY and prove that KY = ½ AD.

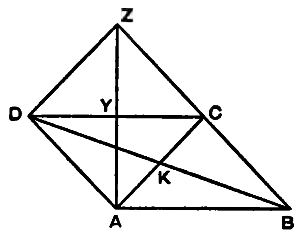


FIG. 772

18. If in fig. 772, ABCD is a parallelogram and if Y is any point on CD, prove that △ DYZ = △ YBC.

19. If in fig. 772, ABCD is a parallelogram and if Y is a point on CD such that ABZD is a cyclic quadrilateral, prove that DZ = AB.

20. With the data of No. 19, prove that DB touches the circle DYZ.

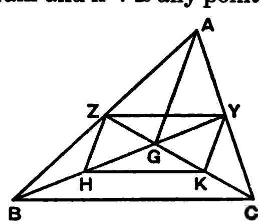


FIG. 773

21. In fig. 773, AY = YC, AZ = ZB, BY cuts CZ at G, GH = HB, GK = KC. Find out all the facts you can about this figure. *Give reasons.*

Nos. 22–31 refer to fig. 774 in which PAQ, RBS, AXY are straight lines.

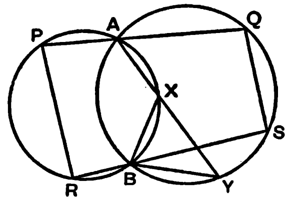


FIG. 774

22. Prove that PR is parallel to QS.

23. What results can be deduced from No. 22 by taking (i) S very close to Q, (ii) S very close to B, (iii) A very close to B?

24. What points in the figure must be joined to give a line parallel to RX? *Give reasons.*

25. If AR is a diameter of circle ABP, prove that AS is a diameter of circle ABQ.

26. If PQSR is a cyclic quadrilateral, prove that PQ = RS.

27. If AP = BX, prove that XP = AB.

28. Prove that $\angle$XBY = $\angle$PBQ.

29. If the circles are equal, prove that BX = BY.

30. Prove that $\angle$XBY is equal to one of the angles between the tangents at A to the two circles.

31. If AR, AS are tangents to the circles ABS, ABR, prove that they are diameters of the circles ABR, ABS.

Nos. 32–35 refer to fig. 775 which represents two circles, centres A, B, radii $a$, $b$ units, touching externally at C. DRTQ, DR′T′Q′, TCT′ are common tangents.

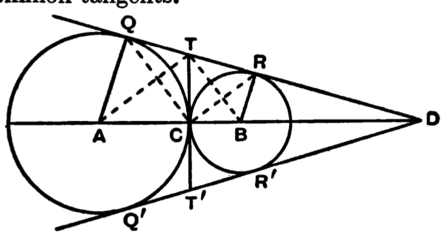


FIG. 775

32. Discover as many facts as you can about fig. 775.

33. Prove that AQ does not equal QR unless $a = 4b$.

34. Prove that RB produced does not touch the circle, centre A, unless $a = 4b$.

35. If RB produced passes through T′, prove that $\triangle$DTT′ is equilateral and that $a = 3b$.

Nos. 36–50 refer to fig. 776 in which ABCD is a cyclic quadrilateral whose opposite sides meet at E and F and whose diagonals meet at G.

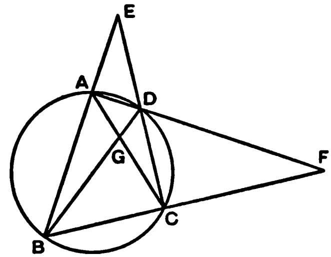


FIG. 776

36. If BD bisects the angles at B and D, prove that $\angle$ BAD is a right angle.

37. What is the relation between the angles EAF, ECF?

38. If $\angle$ BEC $=24°$, $\angle$ AFB $=42°$, $\angle$ BGC $=72°$, find $\angle$ EAD and $\angle$ ACD and $\angle$ DBF.

39. What change must be made in the drawing of the figure if AD, DC, CB are all equal? In this case, if $\angle$ ACB $=\theta$, find $\angle$ CAF and $\angle$ ABC and $\angle$ AFB in terms of $\theta$.

40. Prove that BG > GC. Which is the greater, AG or DG?

41. What change must be made in the drawing of the figure if G is the centre of the circle?

42. Prove that the triangle whose sides are parallel to DA, DB, DC is equiangular to the triangle ABC.

43. Prove that $\angle$ E + $\angle$ F $=180° - 2$ $\angle$ ABC.

44. Prove that the line joining E to the circumcentre of the triangle EAC is perpendicular to BD.

45. Prove that the circumcircles of $\triangle$ EAD and $\triangle$ FCD cut again at a point on EF.

46. Prove that the bisectors of $\angle$ BEC, $\angle$ AFB are at right angles.

47. Prove that the angle between the tangents at A and C is $\angle$ ADC $-$ $\angle$ ABC.

*48. If AC is perpendicular to BD, prove that BC is twice the perpendicular from the centre of the circle to AD.

*49. If AC is perpendicular to BD, prove that the perpendicular from G to AD when produced bisects BC.

*50. Prove that the sum of the squares of the tangents from E and F to the circle is equal to $EF^2$. [Use No. 45.]

## REVISION PAPERS 51–58 (Theorems 1–50)

(*Including angles in a segment*)

### 51

1. ABCD is a square; the bisector of $\angle$ACD cuts BD at Q. Prove that BQ = CD.

2. AD, BC are the parallel sides of the trapezium ABCD; AB = 6 in., BC = 9 in., CD = 5 in., AD = 14 in. Find the area of ABCD.

3. A chord of a circle at a distance 11 cm. from the centre is 6 cm. long. Calculate the length of a chord of the same circle which is at a distance 9 cm. from the centre.

4. O is the centre of a circle; CB is a chord parallel to the radius OA; OB cuts AC at a point K inside the circle. Prove that $\angle$AKB = 3$\angle$ACB.

### 52

1. In $\triangle$ABC, $\angle$ABC = 54°, $\angle$BAC = 78°. If the bisector of $\angle$BCA cuts AB at X, prove that CA = CX.

2. Two straight lines ABC, PQR are cut by three parallel lines AP, BQ, CR. Prove that $\triangle$AQC is equal in area to $\triangle$PBR.

3. ABC is a straight line such that AB = 1 in., BC = 4 in.; PBQ is the chord of the circle on AC as diameter perpendicular to AC. Find the length of PQ.

4. AB is a quadrant of a circle; AC is any chord. If BN is the perpendicular from B to AC, prove that BN = NC.

### 53

1. ABCD is a parallelogram; AB < BC. If the line bisecting $\angle$ABC cuts AD at P, prove that BC = CD + DP.

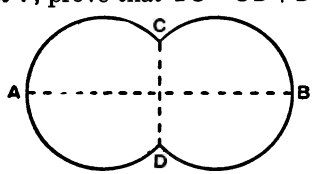



FIG. 777

2. The radii of the circular arcs in fig. 777 are equal and their centres lie on AB. If AB = 10 cm., CD = 4 cm., calculate the radius of each arc.

3. ABCD is a quadrilateral inscribed in a circle, centre O; $\angle ABD = 36°$, $\angle BDC = 43°$, $\angle COD = 40°$; calculate $\angle AOB$.

4. Two circles ABPH, ABQK intersect at A and B. If PBQ is a straight line and if PH is parallel to QK, prove that HAK is a straight line.

## 54

1. ABCD is a parallelogram; ABHK, ADPQ are squares outside ABCD. Prove that QK is equal to one of the diagonals of ABCD.

2. Construct a parallelogram of area 21 sq. cm. such that one side is 6 cm. and one angle is 50° Measure the other side.

3. AB is the diameter of a semicircle APQB; parallel lines PC, QD cut AB at points C, D between A and B. If $AC = DB$, prove that $\angle CPQ$ is a right angle.
What special case is obtained by making Q coincide with B?

4. AB is a chord of a circle, centre O, such that $\triangle OAB$ is equilateral. The line which bisects $\angle OAB$ cuts the circle again at Q. Prove that $AB = BQ$.

## 55

1. ABCD is a square; any line is drawn through A outside the square; BH, DK are the perpendiculars from B, D to this line. Prove that (i) $\triangle ABH \equiv \triangle DAK$; (ii) $BH + DK = HK$.

2. In $\triangle ABC$, $AB = 6$ cm., $AC = 8$ cm., $\angle BAC = 90°$; D is the mid-point of BC. Find (i) the area of $\triangle ABD$, (ii) the length of the perpendicular from B to AD.

3. ABCD is a quadrilateral inscribed in a circle; AC is a diameter. If $\angle BAC = 43°$, find $\angle ADB$.

4. AB is a diameter of a circle; PQ is any chord. The perpendicular from A to PQ is produced to any point H. Prove that the perpendicular bisector of PQ bisects BH.

## 56

1. APRT, AQSC are straight lines such that
$AP = PQ = QR = RS = ST$. Prove that $\angle CST = 5 \angle AQP$.

2. A segment of a circle is cut off by a chord of length 6 cm.; the height of the segment is 2 cm. Calculate the radius of the circle.

3. In fig. 778, A, B are the centres of the circles CDQ, CPD; CPQ is a straight line and BP cuts AQ at K. Prove that $\angle AKB = \angle ACB$.

4. AB, AC are equal chords of the circle ABC and AP, BQ are parallel chords. Prove that $\angle PBQ$, $\angle ABC$ are either equal or supplementary.

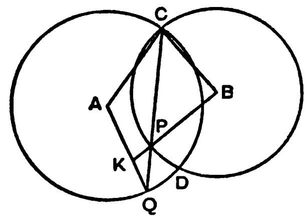



FIG. 778

## 57*

1. ABC is an equilateral triangle; P, Q, R are points on BC, CA, AB such that PQR is also an equilateral triangle. Prove that (i) $\triangle BPR \equiv \triangle CQP$, (ii) $AQ + AR = BC$.

2. Draw a quadrilateral ABCD such that $AB = 6$ cm., $BC = 5$ cm., $CD = 4$ cm., $\angle ABC = 110°$, $\angle BCD = 95°$. Reduce it to an equivalent triangle with AB as base and vertex on BC. Find its area.

3. AB is the diameter of the semicircle AQRB; RQ and BA when produced meet at P. If $\angle QBR = 36°$ and $\angle APQ = 20°$, calculate $\angle RQB$.

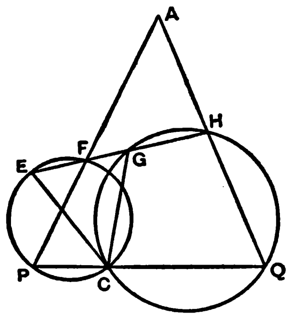



FIG. 779

4. In fig. 779, EFGH is a straight line cutting the circles CFEP, CGHQ; PCQ is a straight line and PF, QH meet at A. If $AP = AQ$, prove that $CE = CG$.

## 58*

1. In the quadrilateral ABCD, $\angle DAB = \angle ABC = 60°$ and $\angle ADC = 90°$. Prove that $AB + BC = 2AD$.

2. E, F are the mid-points of the sides AC, AB of $\triangle ABC$. If BE cuts CF at G, prove that the triangles BGC, CAG, ABG are equal in area.

3. Two parallel chords AB, CD of a circle are 1 in. apart; $AB = 4$ in., $CD = 6$ in. Find the radius of the circle correct to $\frac{1}{10}$ in.

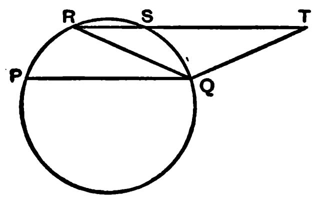



Fig. 780

4. In fig. 780, PQ, RS are parallel chords of the circle; RS is produced to T so that $QR = QT$. Prove that $ST = PQ$.

## REVISION PAPERS 59–66 (Theorems 1-57)

[*Including tests for concyclic points, equal arcs*]

## 59

1. ABCD is a parallelogram, BCH, DCK are equilateral triangles outside it. Prove that (i) $\triangle ADK \equiv \triangle HBA$; (ii) if AD is produced to E, $\angle EDK = \angle DAB - \angle KAH$; (iii) $\angle KAH = 60°$; (iv) $KA = KH$.

2. ABCD is a rectangle, 3 in. by 4 in; a halfpenny (diameter 1 in.) is in the plane of the rectangle and is made to roll once completely round the outside of it. Find the distance travelled by its centre.

3. O is the centre of the circle circumscribing $\triangle ABC$; AD is an altitude of $\triangle ABC$. If AO bisects $\angle BAD$, prove that $\angle ACB - \angle ABC = \frac{1}{3}\angle BAC$.

4. ABCD is a quadrilateral, right-angled at B, C; a line perpendicular to AD cuts AD, BC at P, Q. Prove $\angle BPC = \angle AQD$.

## 60

1. The side BC of an equilateral triangle ABC is produced to D so that $CD = 3BC$. Prove that $AD^2 = 13AB^2$.

2. ABCD is a quadrilateral. If $\angle ABC + \angle ADC = 180°$, prove that the perpendicular bisectors of AC, BD, AB are concurrent.

3. In fig. 781, A is the centre of the circle BCP; PBQ is a straight line. Prove that $QP = QC$.
What can you say about the position of the centre of the circle BCQ?

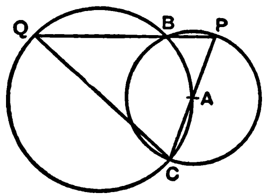


FIG. 781

4. ABCD is a rectangle; the line through C perpendicular to AC cuts AB, AD produced at P, Q. Prove that the points P, B, D, Q are concyclic.

## 61

1 H, K, L are the mid-points of the sides AB, AC, BC of $\triangle ABC$; P, Q are points on BC such that $BP = \frac{1}{4}BC = \frac{1}{3}BQ$. Prove that (i) PH is parallel to AL; (ii) $PH = QK$.

2. ABCD is a quadrilateral such that $\angle BAD = 127°$, $\angle BCD = 53°$, $\angle ABD = 31°$. Calculate $\angle ACB$.

3. AB, AC are equal chords of the circle ABDC, and $BD = BC$. If BA, DC when produced meet at K, and if AD cuts BC at X, prove that $\triangle CAK$, $\triangle XCD$ are equiangular.

4. A, B, C, P, Q, R are points on the circle ABPCQR such that $\angle ABC$ and $\angle PQR$ are right angles. Prove that AP is equal and parallel to CR.

## 62

1. ABCD is a square; P is a point on AB such that $AP = \frac{1}{3}AB$; Q is a point on PC such that $PQ = \frac{1}{3}PC$. Prove $APQD = \frac{1}{3}ABCD$.

2. The side AB of a cyclic quadrilateral ABCD is produced to E; $\angle DBE = 140°$, $\angle ADC = 100°$, $\angle ACB = 45°$. Find $\angle BAC$, $\angle CAD$.

3. AB, AC are equal chords of a circle; BC is produced to D so that $CD = CA$; DA cuts the circle again at E. Prove that BE bisects $\angle ABC$.

4. OY is the bisector of $\angle XOZ$; P is any point; PX, PY, PZ are the perpendiculars from P to OX, OY, OZ. Prove that $XY = YZ$. [Draw the circle on OP as diameter.]

## 63

1. In fig. 782, PN is perpendicular to AC, and PR is parallel to AC; also $QR = 2AP$. Prove that $\angle CAR = \frac{1}{3}\angle CAB$. [If K is the mid-point of QR, $KP = KQ = KR$.]

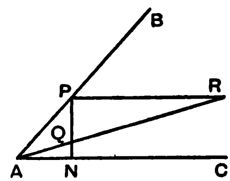


FIG. 782

2. In the quadrilateral ABCD, $AB = 5$ in., $BC = 12$ in., $CD = 7$ in., $\angle ABC = \angle BCD = 90°$. If P is a point on BC such that $\angle APD = 90°$, calculate the length of BP. [Two answers.]

3. PAB, PBC, PCA are three unequal circles. From any point D on the circle PBC, lines DB, DC are drawn and produced to meet the circles PBA, PCA again at X, Y. Prove that XAY is a straight line.

4. AB, BC, CD are equal chords of the circle ABCD; AD is produced to E so that $DE = DC$. Prove that (i) $CE = CA$; (ii) CE is parallel to BD.

## 64

1. In $\triangle ABC$, $\angle ACB = 90°$ and $AC = 2CB$; CD is an altitude. Prove by using the figure of Pythagoras' theorem, p. 282, that $AD = 4DB$.

2. Two chords AB, DC of a circle, centre O, are produced to meet at E; $\angle CBE = 75°$, $\angle CEB = 22°$, $\angle AOD = 144°$. Prove that $\angle AOB = \angle BAC$.

3. In fig. 783, prove that QR is parallel to ST.

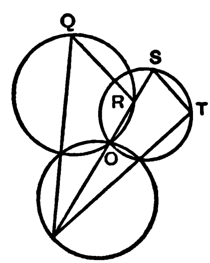


FIG. 783

4. Five points A, B, C, D, E are taken in order on a circle so that the chords AB, AE are equal. If AC, AD meet BE at X, Y, prove that C, X, Y, D are concyclic. [Join CE, CD.]

## 65*

1. ABCD is a quadrilateral such that AB = 7 in., CD = 11 in., ∠BAD = ∠ADC = 90°, ∠BCD = 60°. Calculate AC. [If BN is the perpendicular from B to CD, BC = 2NC, why?]

2. A and B are the centres of the circles CDP, CDQ; PCQ is a straight line. Prove ∠APD = ∠BQD. [Join DA, DB, DC.]

3. ABCD is a circle. If arc ABC = $\frac{1}{2}$ arc ADC, find ∠ADC.

4. The side CD of the square ABCD is produced to E; P is any point on CD; the line from P perpendicular to PB cuts the bisector of ∠ADE at Q. Prove that BP = PQ. [Prove that B, P, D, Q are concyclic.]

## 66*

1. In fig. 784, PR is equal and parallel to BA; PQAT and CQRS are parallelograms, prove that their areas are equal. [Join AR, AP, RC.]

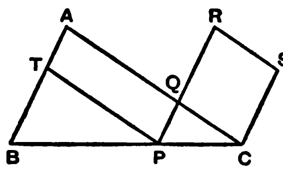


FIG. 784

2. ANB is a diameter of a circle perpendicular to a chord PNQ; AN = $h$, PQ = $a$; find AB in terms of $a$, $h$.

3. ABCD is a cyclic quadrilateral; AB, DC meet when produced at E; AD, BC meet when produced at F. If ∠BEC = 20° and ∠CFD = 40°, find ∠DAB.

4. The centre C of the circle AHBK lies on the circle ACBP; AHP and PBK are straight lines. Prove that (i) AH = BK; (ii) HK = AB. [Join CA, CH, CB, CK.]

# REVISION PAPERS 67–74 (Theorems 1–63)

[*Including tangent properties.*]

## 67

1. In fig. 785, AB is a diameter; ∠HPQ = ∠KQP = 90°. Prove that AH = BK. [Draw the perpendicular from the centre to PQ.]

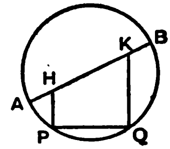


FIG. 785

2. AOB is a chord of a circle ABC; T is a point on the tangent at A; the tangent at B meets TO produced at P; ∠ATO = 35°, ∠BOT = 115°. Find ∠BPT.

3. APC is a minor arc of a circle, centre O; AQOC is another circular arc. Prove that $\angle APC = \angle PAQ + \angle PCQ$.

4. AB is a chord of a circle; AT is the tangent at A; AC is a chord bisecting $\angle BAT$. Prove that $AC = CB$.

## 68

1. ABCD is a parallelogram; P is any point on CD; PB, CB, AD cut any line parallel to AB at Y, Z, W. Prove that quad. $DCZW = 2\triangle APY$.

2. AB, CD are two intersecting chords of a circle; AP, CQ are the perpendiculars from A, C to CD, AB respectively. Prove that PQ is parallel to BD. [Join AC.]

3. The radii of two circles are 2 cm., 5 cm. and the distance between their centres is 9 cm. Calculate the lengths of the interior and exterior common tangents.

4. AB is a diameter of a circle; AC is any chord; P is the mid-point of the arc BC. Prove that AC is perpendicular to the tangent at P.

## 69

1. In $\triangle ABC$, $\angle ABC = 90°$, $\angle BAC = 45°$. The bisector of $\angle ACB$ meets AB at P. Prove that $AP^2 = 2PB^2$. [Draw PN perpendicular to AC.]

2. A circle is drawn touching the sides BC, CA, AB of $\triangle ABC$ at X, Y, Z. If $\angle B = 36°$ and $\angle C = 66°$, calculate $\angle YXZ$.

3. The circle ABPC passes through the centre C of the circle ABQ; APQ is a straight line. Prove that (i) $PB = PQ$; (ii) CP produced bisects BQ at right angles. [Join AC, BC.]

4. AOB is a diameter of a circle, centre O. The tangent at B meets any chord AP produced at T. Prove $\angle ATB = \angle OPB$.

## 70

1. ABCD is a parallelogram; AB, CB are produced to X, Y; P is any point within the angle XBY. Prove that $\triangle PCD - \triangle PAB = \triangle ABC$.

2. In fig. 786, AP, AQ are tangents to the circles ABQ, ABPR; QBR is a straight line. Prove that RP is parallel to AQ.

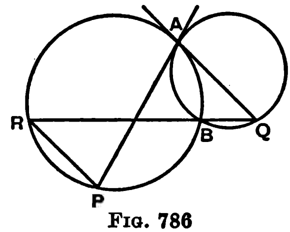



FIG. 786

3. **ABCD** is a square inscribed in a circle; **P** is any point on the minor arc **AB**. Prove that

$$\angle APB = 3\angle BPC.$$

4. **DE** is a diameter of the circle **ADBC**, perpendicular to the chord **BC**. If **AB** > **AC**, prove that

$$\angle DBA = \tfrac{1}{2}(\angle ACB - \angle ABC).$$

## 71

1. **ABC** is a triangle; **APQB**, **AXYC** are squares outside $\triangle$**ABC**. Prove that **PC** is perpendicular to **BX**.

2. **PR** is a chord of a circle, centre **O**. **T** is a point on the tangent at **P** and **OT** cuts **PR** at **Q**. If **TP** = **TQ**, prove that $\angle$**ROT** = 90°

3. **ABCD** is a quadrilateral inscribed in a circle, centre **O**. If **AC** bisects $\angle$**BAD**, prove that **OC** is perpendicular to **BD**.

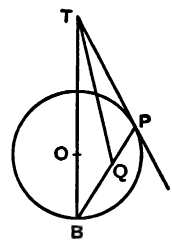


FIG. 787

4. In fig. 787, **TP** is a tangent to the circle, centre **O**, and **TQ** bisects $\angle$**OTP**. Prove that $\angle$**TQP** = 45°.

## 72

1. **ABCD** is a parallelogram; **P** is the mid-point of **AD**; **AB** is produced to **Q** so that **AB** = **BQ**. Prove that

$$ABCD = 2\triangle PQD.$$

2. **AB** is a chord of a circle, centre **O**, such that $\angle$**AOB** is obtuse; **BC** is a chord parallel to **OA**. If $\angle$**OAB** = $x$°, find in terms of $x$ the acute angle which **BC** makes with the tangent at **B**.

3. In $\triangle$**ABC**, $\angle$**BAC** = 90° and **AB** < **AC**; **D** is the mid-point of **BC**. A circle touches **BC** at **D**, passes through **A** and cuts **AC** again at **E**. Prove that arc **AD** = 2 arc **DE**.

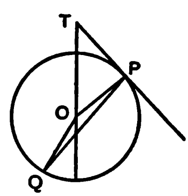


FIG. 788

4. In fig. 788, **TP** is a tangent to the circle, centre **O**; **PQ** and **PT** are equally inclined to **TO**. Prove that $\angle$**QOT** = 3$\angle$**POT**.

## 73*

1. ABCD is a quadrilateral; AB is parallel to CD; BP, CP are drawn parallel to AC, AD respectively to meet at P. Prove that $\triangle PDC = \triangle ABD$.

2. In $\triangle ABC$, $AB = AC$; D is the mid-point of BC. Prove that the tangent at D to the circle ADC is perpendicular to AB. [Join D to A.]

3. Two circular cylinders of radii 2 in., 6 in. are bound tightly together with their axes parallel by an elastic band. Find the stretched length of the band.

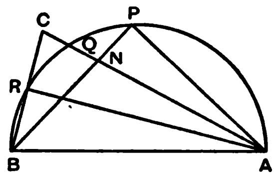



FIG. 789

4. In fig. 789, AB is the diameter of the semicircle APQRB; ANQC is a straight line such that $AB = AC$ and $NQ = QC$. Prove that $\angle BAR = \angle RAQ = \angle QAP$. [Join BQ.]

## 74*

1. ABP, ABQ are equivalent triangles on opposite sides of AB; PR is drawn parallel to BQ to meet AB at R. Prove that QR is parallel to PB.

2. ABC is a triangle inscribed in a circle; BE, CF are altitudes of $\triangle ABC$. Prove that EF is parallel to the tangent at A.

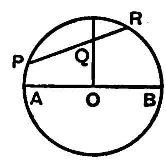



FIG. 790

3. OBC is a straight line such that $OB = 9$ in., $BC = 8$ in.; OA is drawn perpendicular to OB. Calculate the radius of the circle which touches the circle, centre C, radius CB, and touches OB at a point between O and B and also touches OA.

4. In fig. 790, O is the centre of the circle; $PQ = AO$, $\angle AOQ = 90°$. Prove that arc $BR = 3$ arc AP.

## REVISION PAPERS 75–80 (Theorems 1–69)

[*Including extensions of Pythagoras and segments of chords.*]

## 75

1. If each diagonal of a quadrilateral bisects the area of the quadrilateral, prove that the quadrilateral is a parallelogram.

2. ABC is a triangle inscribed in a circle; a line parallel to AC cuts BC at P and cuts the tangent AT at T. Prove that $\angle APC = \angle BTA$. [What quadrilateral must you prove cyclic ?]

3. ABC is an equilateral triangle; D is the mid-point of BC, E is the mid-point of CD. Prove that $AE^2 = 13\,EC^2$.

4. From a point X outside a circle, a line XAB is drawn cutting the circle at A, B; XT is the tangent from X to the circle. If AB = 12 in. and AX = 4 in., find the length of XT.

## 76

1. Draw a figure to illustrate that $(a+b)^2$ is not equal to $a^2+b^2$ unless $a$ or $b$ is zero.

2. PQ, PR are equal chords of a circle; PQ and the tangent at R meet at T. Prove that $\angle PRQ = 60° \pm \frac{1}{3}\angle PTR$. [T may be either on QP produced or on PQ produced.]

3. PQR is a triangle inscribed in a circle. If PQ = PR = 15 cm. and QR = 18 cm., calculate the diameter of the circle. Find also the diameter from an accurate scale-drawing.

4. CA, CB are two fixed radii of a given circle; P is a variable point on the circumference of the circle; PQ, PR are the perpendiculars from P to CA, CB. Prove that QR is of constant length. [Draw the circle on CP as diameter.]

## 77

1. Find the value of $k$ if the points whose co-ordinates are $(1, k)$, $(3, 1)$, $(5, k)$, $(3, 5)$ are the vertices of a square.

2. Draw $\triangle ABC$ so that BC = 9 cm., CA = 8 cm., AB = 5 cm., and take a point D on BC so that BD = 7 cm. Construct a point X on BA produced such that $\triangle DBX = \triangle CBA$. Measure BX.

3. The altitudes AD, BE, CF of an acute angled triangle ABC meet at H. Prove that (i) $\angle FDH$ is the complement of $\angle BAC$; (ii) if K is the circumcentre of $\triangle AFE$, D, F, K, E are concyclic.

4. X is a point distant 7 cm. from the centre O of a circle of radius 9 cm.; AXB is a chord of the circle such that AX = 2XB. Find the length of AB.

## 78

1. ABCD is a quadrilateral in which AD is parallel to BC. If P is the mid-point of CD, prove that $\triangle APB = \frac{1}{2}$ quad. ABCD. [Through P draw XPY parallel to BA to cut BC, AD at X, Y.]

2. The tangents at points P, Q on a circle meet at T; points H, K are taken on TQ and TQ produced respectively so that PQ bisects $\angle$HPK. Prove that $\angle$HPT = $\angle$PKT.

3. In $\triangle$ABC, AB = AC = 9 cm., BC = 12 cm.; BC is trisected at Y and Z. Calculate the length of AY.

4. ABC is a triangle inscribed in a circle; the tangent at C meets AB produced at K. If BK = $\frac{2}{3}$CK, prove that BK = $\frac{4}{5}$AB.

## 79

1. The co-ordinates of A, B, C are respectively (3, 4), (2, 1), (5, 2). ABCD is a parallelogram, and AC cuts BD at K. Find the co-ordinates of (i) K, (ii) D.

2. BE, CF are altitudes of the acute-angled triangle ABC; Z is the mid-point of AB. Prove that $\angle$ZEF = $\angle$ABC $\sim$ $\angle$BAC. What is the corresponding result if $\angle$BAC is obtuse?

3. TP, TQ are the tangents from a point T to a circle; N is the mid-point of the chord PQ; H is the mid-point of TQ. If PH cuts the circle at R, prove that Q, N, R, H are concyclic.

4. Draw a straight line XY and construct two points A, B on the same side of XY such that the distances of A, B from XY are 1 in., $2\frac{1}{2}$ in. respectively and AB = 3 in. Construct the smaller of the two circles which pass through A and B and touch XY.

## 80

1. ABCD is a quadrilateral in which AB is parallel to DC. Prove that $AC^2 + BD^2 = AD^2 + BC^2 + 2AB \,.\, DC$. [Draw DH, CK perpendicular to AB.]

2. ABC is a triangle inscribed in a circle; K is any point inside $\triangle$ABC; AK, BK, CK when produced meet the circle again at X, Y, Z respectively. Prove that $\angle$YXZ = $\angle$BKC − $\angle$BAC.

3. In $\triangle$ABC, BC = 26 cm., CA = 17 cm., AB = 21 cm.; D is the mid-point of BC; E is the mid-point of AD. Calculate the lengths of AD and BE.

4. In $\triangle$ABC, CA = 5 in., CB = 12 in., $\angle$ACB = 90°; a circle, centre O, radius 9 in., is drawn touching CB at B and on the same side of CB as A. If the circle cuts AB again at D, calculate the lengths of AO and BD.

# *PART III*

## SIMILAR FIGURES

### Ratio

If the lengths of two straight lines are 4 cm. and 6 cm., the length of the first is $\frac{4}{6}$ or $\frac{2}{3}$ of that of the second, and we say that the *ratio* of the length of the first line to that of the second is 2 to 3, written 2 : 3, and this ratio is repesented by the fraction $\frac{2}{3}$.

Ratios should be expressed as simply as possible; just as the fraction $\frac{20}{25}$ is equivalent to $\frac{4}{5}$, so the ratio 20 : 25 is equivalent to 4 : 5. A ratio is unaltered if the two numbers or quantities in the ratio are both multiplied, or both divided, by the same number.

A ratio is a comparison of the magnitudes of two quantities which must be *of the same kind*; it is meaningless to compare 5 ounces with 10 shillings or to compare 6 inches with 4 sq. inches.

If two quantities have a common measure, their ratio can be expressed as the ratio of two integers, *e.g.* if the lengths of two straight lines are given to be 2·56 in., 1·12 in., since $\frac{2\cdot56}{1\cdot12}=\frac{256}{112}=\frac{16}{7}$, the ratio of their lengths is 16 : 7. Here the common measure may be taken as $\frac{1}{100}$ in. But we frequently meet with pairs of lines whose lengths have no common measure: if the side of a square is 1 in., the diagonal is $\sqrt{2}$ in. (*Pythagoras*), and these two lengths have no common measure and are called **incommensurable.** The ratio of two such lengths cannot be expressed as the ratio of two integers, although two integers can be found whose ratio differs from this ratio by an amount as small as we please. Formal proofs of theorems involving the ratio of incommensurable quantities are very difficult, and we shall assume that if a theorem has been proved for all commensurable ratios, it is also true if the ratios are incommensurable.

## Ratio of Segments of a Line

If P is any point on a straight line AB or on AB produced or on BA produced, PA and PB are called *segments* of the line AB, see p. 421, and the line AB is said to be divided at P in the ratio AP : PB.

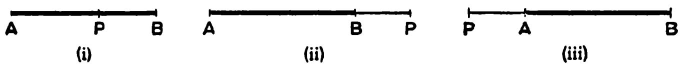



FIG. 791

If P lies between A and B, see fig. 791 (i), the line AB is said to be **divided internally** at P in the ratio AP : PB.

If P lies on AB produced or on BA produced, see fig. 791 (ii), (iii), the line AB is said to be **divided externally** at P in the ratio AP : PB.

It is important to notice that in all cases the *whole* line AB does not appear in the ratio AP : PB of the segments of AB. This definition may also be emphasised, if considered advisable, by a discussion of directed lengths and the interpretation of positive and negative ratios.

## Proportion

If four quantities $a$, $b$, $c$, $d$ are such that

$$a : b = c : d$$

then $a$, $b$, $c$, $d$ are said to be **in proportion.**

Thus if $a$, $b$, $c$, $d$ are in proportion, we have $\frac{a}{b} = \frac{c}{d}$, and $d$ is called the **fourth proportional** to $a$, $b$, $c$.

If three quantities $a$, $b$, $c$ are such that

$$\frac{a}{b} = \frac{b}{c}$$

then $a$, $b$, $c$ are said to be in **continued proportion.** Further, $c$ is then called the **third proportional** to $a$, $b$; and $b$ is called a **mean proportional** between $a$, $c$.

Thus, if $b$ is a mean proportional between $a$, $c$,

$$b^2 = ac.$$

Therefore, if a square, side $b$ units, is equal in area to a rectangle whose adjacent sides are $a$ units, $c$ units, then $b$ is a mean proportional between $a$ and $c$.

### Examples for Oral Discussion

1. If $\frac{a}{b}=\frac{c}{d}$, prove that $\frac{a}{b}=\frac{2a-3c}{2b-3d}$.

If $\frac{a}{b}=\frac{c}{d}=k$, then $a=bk$, $c=dk$.

Substitute for $a$ and $c$.

2. If $\frac{a}{b}=\frac{c}{d}$, express $\frac{a+b}{a-b}$ in terms of $c$ and $d$.

3. The line AB is divided internally at P in the ratio $x:y$; express in terms of $x$ and $y$, (i) PB : AB; (ii) AB : AP.

4. If, in fig. 792, $\frac{AP}{PB}=\frac{AQ}{QC}$,

(i) prove that $\frac{AP}{AB}=\frac{AQ}{AC}$;

(ii) find a ratio equal to $\frac{AB}{PB}$.

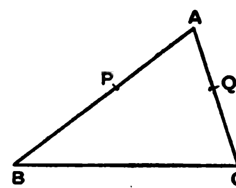


FIG. 792

5. AB is divided internally at X in the ratio 5 : 6 and is divided externally at Y in the ratio 5 : 3. Is X nearer to A or B? Is Y nearer to A or B?

6. Are the following in proportion:

(i) $3\frac{1}{3}$, 5, 8, 12; (ii) 8 in., 6 deg., 12 deg. 9 in.?

## EXERCISE 83

Express the following ratios as simply as possible:

1. 9 in. : 2 ft. [2] 3 rt. $\angle$s : 120°. 3. 12 sq. ft. : 2 sq. yd.

Find the values of $x$ in Nos. 4–7.

4. $3:x=4:10$. [5] $(x-5):(x+5)=3:7$.

6. $6:x=x:24$. [7] $x$ ft. : 5 yd. $=2:3$.

Express in the form of equal ratios the relations, Nos. 8–10.

**8.** $pq = xy$. [9] XA . XB = XT$^2$. **10.** rect. ABCD = rect. PQRS.

**11.** A line AB, 8 in. long, is divided internally at P in the ratio 2 : 3. Find AP.

**12.** A line AB, 8 in. long, is divided externally at Q in the ratio 7 : 3. Find AQ.

[13] A line AB, 6 in. long, is divided externally at R in the ratio 2 : 7. Find AR.

[14] A line AB, 12 cm. long, is divided internally at P in the ratio 3 : 5, externally at Q in the ratio 4 : 9, and externally at R in the ratio 8 : 3. Find the lengths of PQ and PR.

[15] A line AB, 8 in. long, is divided internally at C and externally at D in the ratio 9 : 5; O is the mid-point of AB. Prove that OC . OD = OB$^2$.

**16.** A line AB, 6 in. long, is divided internally at C and externally at D in the ratio 4 : 1; O is the mid-point of CD. Find the ratio AO : BO.

**17.** A line AB, 6 in. long, is divided internally at P in the ratio 2 : 1 and externally at Q in the ratio 5 : 2. Find the ratios in which PQ is divided by A and by B.

**18.** A line AB is divided internally at P and externally at Q in the ratio $c : d$. If AB $= 2b$ in., find the lengths of AP and AQ.

If O is the mid-point of AB, prove that OP . OQ = OB$^2$.

[19] A line AB is bisected at O and divided internally at P in the ratio $x : y$. Find the ratio OP : AB in terms of $x$ and $y$. [Let AB $= 2l$ in.]

**20.** If $\frac{a}{b} = \frac{c}{d}$, state ratios equal to (i) $b : a$; (ii) $a : c$; (iii) $b : d$.

**21.** If $\frac{a}{b} = \frac{c}{d}$, state ratios equal to (i) $(a+b) : b$; (ii) $a : (a+b)$.

If $\frac{a}{b} = \frac{c}{d}$, prove the relations in Nos. 22–24.

**22.** $\dfrac{b}{a-b} = \dfrac{d}{c-d}$. **23.** $\dfrac{a+c}{b+d} = \dfrac{a-c}{b-d}$. [24] $\dfrac{a^2}{b^2} = \dfrac{a^2-c^2}{b^2-d^2}$.

If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, fill up the blank spaces in Nos. 25–27.

25. $\frac{a}{b}=\frac{\cdots}{b+d+f}$.  [26] $\frac{a+c}{b+d}=\frac{c-e}{\cdots}$.  27. $\frac{a-3c}{b-3d}=\frac{2a+7c-5e}{\cdots}$.

28. ABCDE is a straight line such that

$$AB : BC : CD : DE = 1 : 3 : 2 : 5.$$

Find the ratios (i) AB : AE; (ii) AC : CE; (iii) EB : AD.
Find also the ratios in which BE is divided internally by D and externally by A. If BE = 4 in., find AC.

29. ABCDEF is a straight line such that

$$AB : BC : CD : DE : EF = p : q : r : s : t.$$

Find the ratios (i) AB : AF; (ii) BE : CF.
Find also the ratios in which CF is divided externally by A and internally by E. If BD $=x$ in., find AE.

[30] ABC is a straight line. If AC : AB $=n : 1$, find AB : BC.

[31] AB is divided internally at C and externally at D in the ratio $x : y$. Find (i) the ratio CD : AB, (ii) the ratio in which B divides CD.

32. ABCD, AXYZ are two straight lines such that AB : BC : CD = AX : XY : YZ. Fill up the blank spaces in the following: (i) $\frac{AB}{AX}=\frac{AC}{\cdots}$; (ii) $\frac{BC}{AD}=\frac{\cdots}{AZ}$; (iii) $\frac{XZ}{AY}=\frac{\cdots}{AC}$.

## EQUAL RATIOS

### Examples for Oral Discussion

1. In fig. 793, AB is divided internally at X in the ratio 3 : 5, and XY is drawn parallel to BC to meet AC at Y. Prove that AC is divided at Y in the same ratio 3 : 5. Find the values of the ratios AX : AB and AY : AC?

If AB is divided into 8 equal parts, AX contains 3 of these parts and XB contains 5 of them.

Use the intercept theorem.

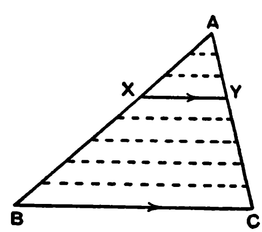



FIG. 793

2. The side AB of $\triangle$ABC is divided internally at P in the ratio 4 : 7, and PQ is drawn parallel to BC to meet AC at Q. What construction can you use to prove that AC is divided at Q in the same ratio, 4 : 7?

What are the values of the ratios PB : AB and QC : AC?

3. Mark on *squared paper* the points A (0, 0), B (3, 0), C (2, 3), the unit being 1 in., and draw $\triangle$ABC. Draw the line joining (0, 1·6) to (3, 1·6) and let it cut CA, CB at X, Y.

Into how many equal parts do the printed lines parallel to AB divide CX and XA?

What are the values of the ratios

(i) $\frac{CX}{XA}$, $\frac{CY}{YB}$; (ii) $\frac{CX}{CA}$, $\frac{CY}{CB}$?

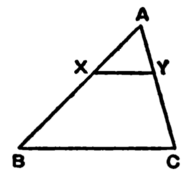


FIG. 794 (i)

4. **If in fig. 794 (i) XY is parallel to BC,**

**prove that $\frac{AX}{XB} = \frac{AY}{YC}$.**

Suppose that $\frac{AX}{XB}$ can be expressed in the form $\frac{p}{q}$, where $p$ and $q$ are integers.

5. If with the data of No. 4, $\frac{AX}{XB} = \frac{p}{q}$, express the ratios $\frac{AX}{AB}$ and $\frac{AY}{AC}$ in terms of $p$ and $q$.

6. What ratio of lengths in fig. 793 is equal to YC : AC?

7. **If in fig. 794 (ii), XY is parallel to BC, prove that**

AX : XB = AY : YC.

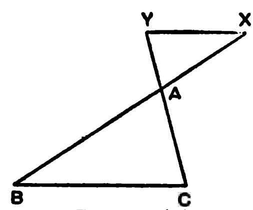


FIG. 794 (ii)

The results established in Nos. 4, 7 may be expressed as follows:—

**A straight line drawn parallel to one side of a triangle divides the other sides proportionally.**

**8. If in fig. 795, $AX : XB = AY : YC$, prove that $XY$ is parallel to $BC$.**

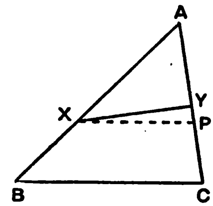


FIG. 795

Let the line through X parallel to BC cut AC at P.

Explain why $AP : AC = AY : AC$. This proves that P is the same point as Y.

**9. If in fig. 794 (ii), $AX : XB = AY : YC$, prove that $XY$ is parallel to $BC$.**

Use the construction and method of No. 8.

The results established in Nos. 8, 9 may be expressed as follows:—

**If two sides of a triangle are divided in the same ratio, both internally or both externally, the straight line joining the points of section is parallel to the third side.**

These results may also be obtained by using the theorem that the area of a triangle is measured by half the product of the base and altitude.

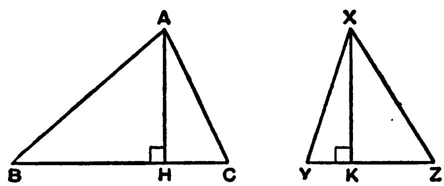


FIG. 796

**10. In fig. 796, the altitudes AH, XK of the triangles ABC, XYZ are equal. Prove that** $\dfrac{\triangle ABC}{\triangle XYZ} = \dfrac{BC}{YZ}$.

11. In fig. 797, BCD is a straight line. Express as the ratio of two lengths,

(i) $\dfrac{\triangle ABC}{\triangle ACD}$; (ii) $\dfrac{\triangle ABC}{\triangle ABD}$.

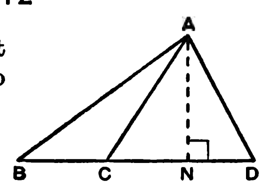


FIG. 797

12. In fig. 798, express as the ratio of two lengths,

(i) $\dfrac{\triangle AXY}{\triangle BXY}$; (ii) $\dfrac{\triangle XAY}{\triangle XCY}$.

What can you say about these ratios if XY is parallel to BC?

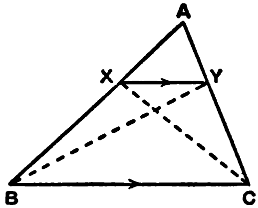



FIG. 798

13. Repeat No. 12 for fig. 799 (i).

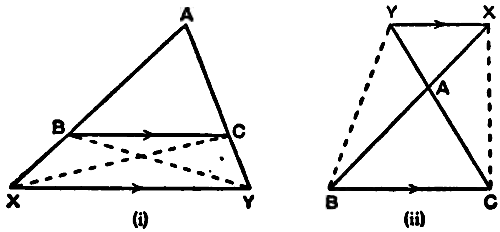



FIG. 799

14. Repeat No. 12 for fig. 799 (ii).

## NUMERICAL EXAMPLES

### EXERCISE 84

[*Arrows indicate that lines are given parallel.*]

Nos. 1–9 refer to fig. 800.

Name *two* ratios equal to each of the ratios in Nos. 1–6:

1. $\dfrac{OA}{AB}$. [2] $\dfrac{OE}{OF}$. 3. $\dfrac{CD}{OD}$.

4. OA : OC. [5] OD : OF. 6. AB : EF.

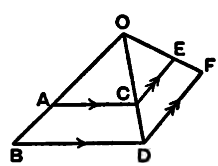



FIG. 800

7. If OA = 10·5 cm., AB = 4·5 cm., OD = 7 cm., find CD.

[8] If OB = 19·5 cm., OA = 12 cm., CD = 6 cm., find OC.

9. If OA = 12 cm., AB = 9 cm., OC = 8 cm., EF = 4·5 cm., find CD and OF.

Nos. 10–13 refer to fig. 801.

Name *two* ratios equal to each of the ratios in Nos. 10–12:

10. AK : KC.

[11] BA : AH.

[12] AP : PQ.

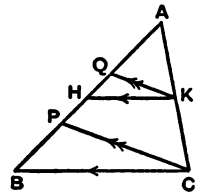


FIG. 801

13. If AQ = 6 cm., QH = 4 cm., HP = 5 cm., KC = 18 cm., find AK and PB.

[14] If in fig. 799 (ii), p. 460, AX = 2 in., BX = 5 in., AC = $2\frac{1}{2}$ in., find CY.

Find the unknown marked lengths in Nos. 15–18, unit 1 cm.:

15.

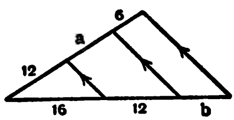


FIG. 802

16.

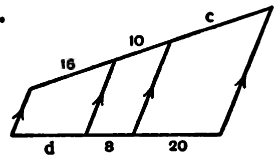


FIG. 803

17.

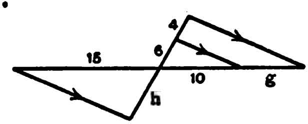


FIG. 804

18.

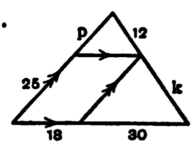


FIG. 805

[19] ACE, BDF are two straight lines cut by three parallel lines AB, CD, EF. If AC = 2 in., CE = 3 in., BF = 4 in., find BD.

[20] P is any point on the side AB of $\triangle$ABC; PB is divided internally at Q in the ratio 2 : 3; PM, QN, BK are the perpendiculars from P, Q, B to AC. If AM = 4 cm., AK = 7 cm., find AN.

[21] The side AB of $\triangle$ABC is divided at X in the ratio 3 : 4; P, Q are points on CA, CB such that XP, XQ are parallel to CB, CA respectively. If XP = 1·5 in., XQ = 1·6 in., find AC and BC.

[*This exercise is continued at the foot of p.* 463.]

## THEOREM 70

**If two triangles are of equal altitude, the ratio of their areas is equal to the ratio of their bases.**

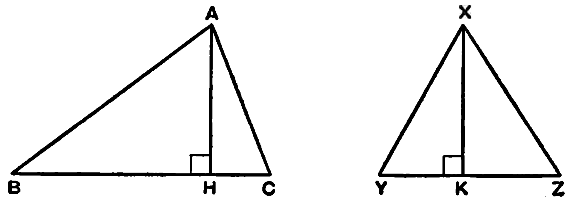



FIG. 806

**Given** two triangles ABC, XYZ in which the altitudes AH, XK are equal.

**To prove that** $\dfrac{\triangle ABC}{\triangle XYZ}=\dfrac{BC}{YZ}$.

**Proof.** The area of a triangle is measured by half the product of the measures of its base and altitude.

$$\therefore \triangle ABC = \tfrac{1}{2} BC \cdot AH$$

and $$\triangle XYZ = \tfrac{1}{2} YZ \cdot XK;$$

$$\therefore \frac{\triangle ABC}{\triangle XYZ} = \frac{\frac{1}{2} BC \cdot AH}{\frac{1}{2} YZ \cdot XK}.$$

But $$AH = XK \qquad \textit{given},$$

$$\therefore \frac{\triangle ABC}{\triangle XYZ} = \frac{BC}{YZ}.$$

NOTE. This proof is only valid if the measures of the altitude and bases are commensurable.

**Corollary. If BCD is a straight line and if A is any point not on this line,**

$$\frac{\triangle ABC}{\triangle ACD}=\frac{BC}{CD} \quad \text{and} \quad \frac{\triangle ABC}{\triangle ABD}=\frac{BC}{BD}.$$

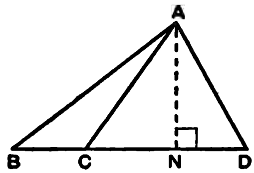



FIG. 807

The perpendicular AN from A to BCD is an altitude of each of the triangles ABC, ACD, ABD.

---

EXERCISE 84 (*continued*)

22. P, R are points on the sides BC, BA of $\triangle$ABC such that BP $=20$ cm., PC $=15$ cm., BR $=16$ cm., RA $=12$ cm. Prove that PR is parallel to CA.

[23] The diagonals of a quadrilateral ABCD cut at K. If AK $=2{\cdot}4$ in., KC $=1{\cdot}6$ in., BK $=1{\cdot}5$ in., KD $=1$ in., prove that AB is parallel to DC.

*24. Draw a quadrilateral ABCD in which $\angle$B, $\angle$C are right angles and AB $=5$ cm., BC $=12$ cm., CD $=4$ cm. Take a point E on BC such that BE $=9$ cm. and join AE, DE. Construct a line cutting AB, AE, DE, DC at P, Q, R, S such that PQ : QR : RS $=4:1:3$.

*25. The side AB of $\triangle$ABC is divided internally at P in the ratio $3:5$; BC is divided internally at R in the ratio $1 . 3$; PQ is drawn parallel to BC to cut AC at Q; RS is drawn parallel to BA to cut PQ, AC at V, S. Find the ratios (i) RV : VS; (ii) PV : VQ.

## THEOREM 71

**If a straight line is drawn parallel to one side of a triangle, it divides the other sides, produced if necessary, proportionally.**

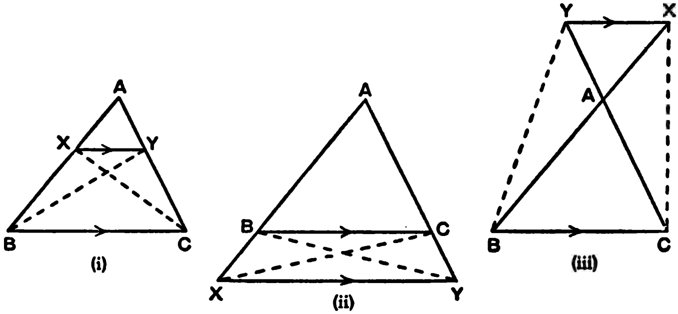



FIG. 808

**Given** a triangle ABC and a line parallel to BC cutting AB, AC, produced if necessary, at X, Y.

**To prove that** $\dfrac{AX}{XB}=\dfrac{AY}{YC}$.

**Construction.** Join BY, CX.

**Proof.** The perpendicular from Y to AB is an altitude of each of the triangles AXY, BXY,

$$\therefore\ \frac{AX}{XB}=\frac{\triangle AXY}{\triangle BXY}.$$

Similarly, $\dfrac{AY}{YC}=\dfrac{\triangle AYX}{\triangle CYX}$.

But $\triangle BXY=\triangle CXY$. *Same base* XY, *and between same parallels* XY, BC.

$$\therefore\ \frac{\triangle AXY}{\triangle BXY}=\frac{\triangle AYX}{\triangle CYX};\qquad \therefore\ \frac{AX}{XB}=\frac{AY}{YC}.$$

**Corollary. If a line XY parallel to BC cuts AB, AC at X, Y,**

**then** $\dfrac{AX}{AB}=\dfrac{AY}{AC}$ and $\dfrac{XB}{AB}=\dfrac{YC}{AC}$.

## THEOREM 72

**If two sides of a triangle are divided in the same ratio, both internally or both externally, the straight line joining the points of section is parallel to the third side.**

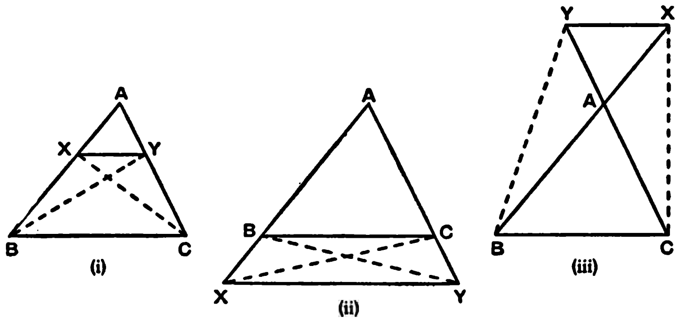



FIG. 809

**Given** a triangle ABC and two points X, Y dividing AB, AC, both internally or both externally, such that

$$\frac{AX}{XB}=\frac{AY}{YC}.$$

**To prove that** XY is parallel to BC.

**Construction.** Join BY, CX.

**Proof.** The perpendicular from Y to AB is an altitude of each of the triangles AXY, BXY,

$$\therefore \frac{AX}{XB}=\frac{\triangle AXY}{\triangle BXY}.$$

Similarly $$\frac{AY}{YC}=\frac{\triangle AYX}{\triangle CYX}.$$

But $\dfrac{AX}{XB}=\dfrac{AY}{YC}$ *given*, $\therefore \dfrac{\triangle AXY}{\triangle BXY}=\dfrac{\triangle AYX}{\triangle CYX}.$

$$\therefore \triangle BXY=\triangle CXY.$$

But these triangles are on the same base XY and on the same side of it,

$$\therefore \text{XY is parallel to BC.}$$

## EXERCISE 85

[*Arrows indicate that lines are given parallel.*]

1. With the data of fig. 800, p. 460, prove that $\frac{OA}{OB}=\frac{OE}{OF}$, and complete the relation $\frac{OF}{EF}=\frac{OB}{\cdots}$. What can you say about the lines AE and BF? Give reasons.

2. With the data of fig. 801, p. 461, prove that $\frac{AQ}{QP}=\frac{AH}{HB}$, and complete the relation $\frac{AB}{BH}=\frac{\cdots}{PQ}$.

[3] Three parallel lines AX, BY, CZ cut two lines ABC, XYZ. Prove that AB : BC = XY : YZ.

4. With the data of fig. 810, prove that QR is parallel to BC. [What ratios must you try to prove equal?]

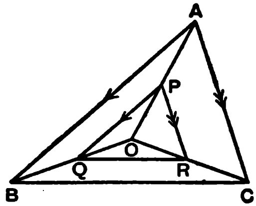


FIG. 810

[5] P is any point on the side AB of the quadrilateral ABCD; PX, PY are drawn parallel to AC, AD to cut BC, BD respectively at X, Y. Prove that XY is parallel to CD. [What ratios must you try to prove equal?]

6. If in fig. 811, BD = EC, prove that PQ is parallel to BC.

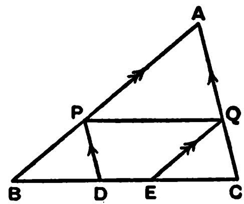


FIG. 811

[7] AB, DC are the parallel sides of a trapezium ABCD; H, K are points on AD, BC such that AH : HD = BK : KC. Prove that HK is parallel to AB. [Draw DPQ || CB to cut HK, AB at P, Q.]

8. ABC is a triangle; P, Q are points on AB, AC such that $AP=\frac{1}{3}AB$ and $CQ=\frac{1}{3}CA$. Prove that the line through C parallel to PQ bisects AB.

9. In fig. 812, PBQ is a straight line. Prove that

PB : BQ = QR : RC.

[10] In fig. 812, PBQ is a straight line. Prove that

PQ : QC = QB : CR.

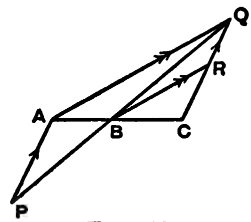


FIG. 812

11. In fig. 813, XABC and XPQ are straight lines. Prove that XA : XB = XB : XC.

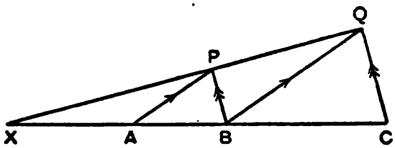



FIG. 813

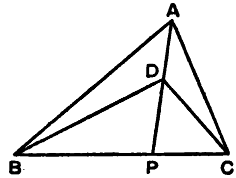



FIG. 814

12. In fig. 814, AD produced cuts BC at P. Prove that

$$\frac{\triangle ABD}{\triangle ACD} = \frac{BP}{PC}.$$

[13] In fig. 814, AD produced cuts BC at P. Prove that

$$\frac{\triangle ABC}{\triangle DBC} = \frac{AP}{DP}.$$

14. A variable straight line RAQP passes through a fixed point A and meets a fixed line BC at P. If $AQ = \frac{1}{3}AP$ and if $RA = QP$, find (i) the locus of Q, (ii) the locus of R.

15. ABC is a triangle; three parallel lines AP, BQ, CR meet BC, CA, AB, produced if necessary, at P, Q, R respectively. Prove that $\frac{BP}{PC} \times \frac{CQ}{QA} \times \frac{AR}{RB} = 1$.

[16] P is any point on the median AD of $\triangle ABC$; AD is produced to Q so that PD = DQ; BP, CP produced meet AC, AB at E, F respectively. Prove that

(i) AP : AQ = AF : AB;
(ii) EF is parallel to BC.

17. In fig. 815, the lines AB, AC and the point D are given. Construct the line PDQ so that PD = 3DQ.

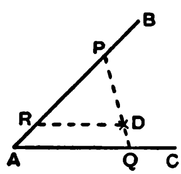



FIG. 815

*18. ABC is a triangle; a straight line cuts BC produced, CA, AB at P, Q, R respectively; CX is drawn parallel to PQ meeting AB at X. Prove that

$$\text{(i) } \frac{BP}{CP} = \frac{BR}{XR}; \quad \text{(ii) } \frac{BP}{CP} \times \frac{CQ}{QA} \times \frac{AR}{RB} = 1.$$

This is called *Menelaus' Theorem.*

*19. In fig. 816, if $AH = HB$ and $AK = 2KC$, prove that

(i) $\triangle BOH = 3\triangle COK$; (ii) $CO = OH$.

*20. In fig. 816, if $HB = \frac{1}{4}AB$ and $KC = \frac{1}{3}AC$, prove that

(i) $BO = OK$; (ii) $CO = 2OH$.

*21. If in fig. 816, $AO$ produced cuts $BC$ at $N$, prove that

$$\frac{BN}{NC} \times \frac{CK}{KA} \times \frac{AH}{HB} = 1. \quad \text{[Use the result in No. 12.]}$$

This is known as *Ceva's Theorem*.

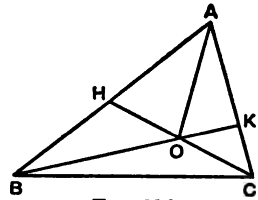


Fig. 816

## Bisectors of an Angle of a Triangle

### Examples for Oral Discussion

**1. In fig. 817, if the internal bisector of $\angle BAC$ cuts $BC$ at $D$, prove that**

$$\frac{BD}{DC} = \frac{BA}{AC}.$$

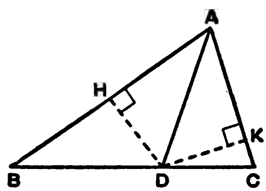


Fig. 817

Draw the perpendiculars $DH$, $DK$ from $D$ to $AB$, $AC$.

(i) Explain why $DH = DK$.

(ii) Express the ratio $\dfrac{\triangle ABD}{\triangle ACD}$ in two different ways.

**2. In fig. 817, if $BC$ is divided internally at $D$ in the ratio $AB : AC$, prove that $AD$ is the internal bisector of $\angle BAC$.**

Use the construction in No. 1 and express $\dfrac{\triangle ABD}{\triangle ACD}$ in two ways.

**3. In fig. 818, if the external bisector of $\angle$BAC cuts BC produced at D, prove that**

$$\frac{BD}{DC}=\frac{BA}{AC}.$$

Draw the perpendiculars DH, DK from D to AB, AC, produced if necessary.

Express the ratio $\frac{\triangle ABD}{\triangle ACD}$ in two different ways.

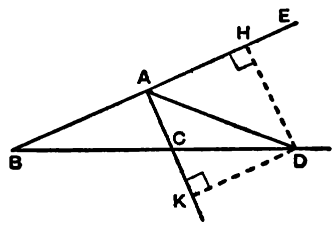



FIG. 818

**4. In fig. 818, if BC is divided externally at D in the ratio AB : AC, prove that AD is the external bisector of $\angle$BAC.**

Use the construction in No. 3 and express $\frac{\triangle ABD}{\triangle ACD}$ in two ways.

5. Construct a line BC of length $\sqrt{8}$ in. Perform the following construction for dividing BC internally and externally in the ratio 5 : 3 and prove that it is correct.

Draw $\triangle$ABC so that AB = 2·5 in., AC = 1·5 in. Draw the internal and external bisectors of $\angle$BAC and let them cut BC at P, Q. Then P and Q are the required points of section.

# NUMERICAL EXAMPLES

## EXERCISE 86

1. In $\triangle ABC$, $BC = 6$ in., $CA = 3$ in., $AB = 5$ in. If the internal and external bisectors of $\angle BAC$ meet $BC$ and $BC$ produced at $X$ and $Y$, find the lengths of $XC$ and $XY$.

[2] In $\triangle ABC$, $BC = 5$ cm., $CA = 4$ cm., $AB = 6$ cm. If the internal and external bisectors of $\angle BAC$ meet $BC$ and $BC$ produced at $P$ and $Q$, find the lengths of $BP$ and $BQ$, and prove that $\frac{1}{BP} + \frac{1}{BQ} = \frac{2}{BC}$.

3. The perimeter of $\triangle ABC$ is 45 in.; the internal bisector of $\angle BAC$ cuts $BC$ at $P$ and the internal bisector of $\angle ACB$ cuts $AB$ at $Q$. If $BP = 9$ in., $CP = 6$ in., find $AQ$.

[4] In $\triangle ABC$, $AB = 4$ in., $BC = 3$ in., $\angle ABC = 90°$. If the bisector of $\angle ACB$ cuts $AB$ at $R$, find $CR$.

5. In $\triangle ABC$, $AB = 12$ cm., $BC = 15$ cm., $CA = 8$ cm.; $P$ is a point on $BC$ such that $BP = 9$ cm. Prove that $AP$ bisects $\angle BAC$. If the external bisector of $\angle BAC$ cuts $BC$ produced at $Q$, and if $D$ is the mid-point of $BC$, prove that

$$DP \cdot DQ = DC^2.$$

[6] If the bisector of $\angle BAC$ cuts $BC$ at $P$, and if the lengths of $BC$, $CA$, $AB$ are $a$, $b$, $c$ units, find the length of $BP$ in terms of $a$, $b$, $c$.

7. The internal and external bisectors of $\angle BAC$ meet $BC$ and $BC$ produced at $P$ and $Q$. If $BP = 5$ in., $PC = 3$ in., find $CQ$.

8. In $\triangle ABC$, $AB = 6$ cm., $AC = 10$ cm., and the bisector of $\angle BAC$ cut $BC$ at $P$. If the area of $\triangle ABC$ is 24 sq. cm., find the area of $\triangle ABP$.

[9] $ABCD$ is a straight line such that $AB = 14$ cm., $BC = 6$ cm., $CD = 15$ cm.; $K$ is a point such that $KA = 21$ cm., $KC = 9$ cm. Prove that $\angle BKD$ is a right angle.

10. A, B are fixed points such that AB $=2\cdot1$ in.; P is a variable point such that PA : PB $=5:2$. If AB is divided internally and externally at C and D in the ratio $5:2$, prove that the locus of P is the circle on CD as diameter. Find the radius of this circle.

[11] ABCD is a rectangular sheet of paper; AB $=4$ in., BC $=3$ in. The edge BC is folded along BD and the corner is then cut off along the crease. Find the area of the remainder.

12. In $\triangle$ABC, AB $=6$ in., AC $=4$ in. The bisector of $\angle$BAC meets the median BE at O. If the area of $\triangle$ABC is 8 sq. in., find the area of $\triangle$AOB.

[13] If I is the in-centre of $\triangle$ABC and if AI cuts BC at P, and if the lengths of BC, CA, AB are $a$, $b$, $c$ units, find the ratio AI : IP in terms of $a$, $b$, $c$.

*14. AD is a median of $\triangle$ABC and the bisector of $\angle$ABC cuts AD at P. If BC $=16$ cm., CA $=11$ cm., AB $=13$ cm., find DP.

*15. APB, CPD are intersecting chords of the circle ACBD and C is the mid-point of the arc AB. If AP $=2\cdot4$ cm., PB $=1\cdot6$ cm., find the ratio DA : DB.

Show how to construct another point E on the circle such that

$$EA : EB = DA : DB.$$

*16. In a circle of radius $1\cdot6$ in. inscribe a triangle ABC such that BC $=2\cdot5$ in. and AB : AC $=4:1$. [Two solutions.]

*17. Construct $\triangle$ABC, given that AB $=4$ in., AC $=2$BC and $\angle$ACB $=120°$.

*18. ABCD is a quadrilateral such that AB $=6$ cm., BC $=8$ cm., CD $=12$ cm., DA $=9$ cm. What can you say about the point of intersection of (i) the bisectors of $\angle$ABC, $\angle$ADC, (ii) the bisectors of $\angle$BAD, $\angle$BCD?

## THEOREM 73 (First Proof)

**If the vertical angle of a triangle is bisected internally or externally by a straight line which cuts the base or the base produced, it divides the base internally or externally in the ratio of the other sides of the triangle.**

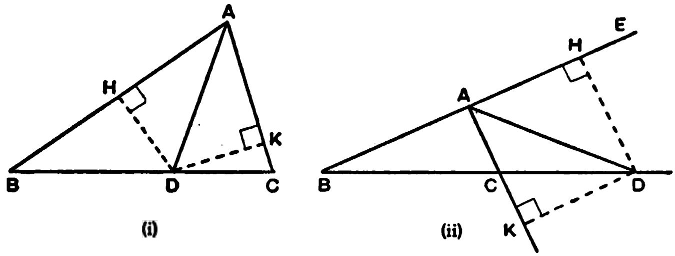



FIG. 819

**Given** a triangle ABC and the line AD which bisects $\angle$BAC internally, fig. 819 (i), or externally, fig. 819 (ii), and cuts BC or BC produced at D.

**To prove that** $\dfrac{BD}{DC}=\dfrac{AB}{AC}$.

**Construction.** From D draw the perpendiculars DH, DK to AB, AC, produced if necessary.

**Proof.** Since D is a point on the bisector of one of the angles formed by AB, AC, D is equidistant from AB and AC.

$\therefore$ DH = DK *locus theorem.*

But DH, DK are altitudes of $\triangle$DAB, $\triangle$DAC,

$$\therefore \frac{AB}{AC}=\frac{\triangle DAB}{\triangle DAC}.$$

Also the perpendicular from A to BC is an altitude of each of the triangles ABD, ADC,

$$\therefore \frac{BD}{DC}=\frac{\triangle ABD}{\triangle ADC}.$$

$$\therefore \frac{BD}{DC}=\frac{AB}{AC}.$$

## THEOREM 74 (First Proof)

**If a straight line through the vertex of a triangle divides the base internally or externally in the ratio of the other sides, it bisects the vertical angle internally or externally.**

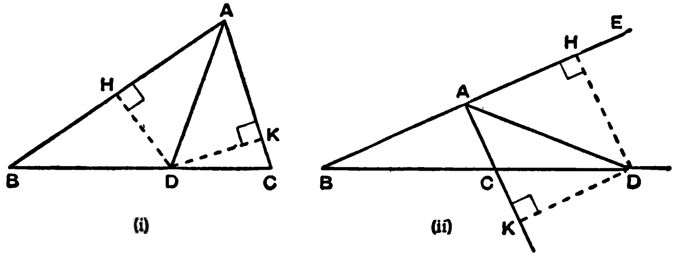



FIG. 820

**Given** a triangle ABC and a point D on BC, fig. 820 (i), or on BC produced, fig. 820 (ii), such that

$$\frac{BD}{DC}=\frac{AB}{AC}.$$

**To prove that** AD bisects $\angle$BAC internally or externally.

**Construction.** From D draw the perpendiculars DH, DK to AB, AC, produced if necessary.

**Proof.** The perpendicular from A to BC is an altitude of each of the triangles ABD, ADC,

$$\therefore \frac{BD}{DC}=\frac{\triangle ABD}{\triangle ADC}.$$

$$\therefore \frac{BD}{DC}=\frac{\frac{1}{2}AB.DH}{\frac{1}{2}AC.DK}=\frac{AB}{AC}\cdot\frac{DH}{DK}.$$

But $$\frac{BD}{DC}=\frac{AB}{AC} \qquad \textit{given},$$

$$\therefore DH = DK.$$

$\therefore$ D is equidistant from the two straight lines AB, AC,

$\therefore$ D lies on one of the bisectors of $\angle$BAC *locus theorem.*

$\therefore$ AD bisects $\angle$BAC internally or externally.

## THEOREM 73 (Second Proof)

**If the vertical angle of a triangle is bisected internally or externally by a straight line which cuts the base or the base produced, it divides the base internally or externally in the ratio of the other sides of the triangle.**

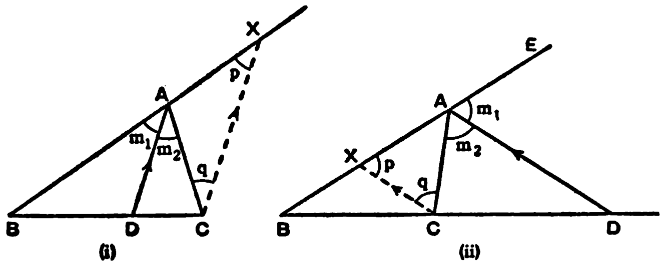



FIG. 821

**Given** a triangle ABC and the line AD which bisects $\angle$BAC internally, fig. 821 (i), or externally, fig. 821 (ii), and cuts BC or BC produced at D.

**To prove that** $\dfrac{BD}{DC}=\dfrac{AB}{AC}$.

**Construction.** Through C draw a line parallel to DA to cut BA, produced if necessary, at X.

**Proof.** With the notation in the figures,

$m_1 = p$ *corr.* $\angle$*s*, DA ∥ CX,

$m_2 = q$ *alt.* $\angle$*s*, DA ∥ CX,

but $m_1 = m_2$ *given*,

$\therefore\ p = q$.

$\therefore$ $\triangle$AXC is isosceles and AX = AC.

Since DA is parallel to CX,

$$\frac{BD}{DC}=\frac{BA}{AX},$$

$$\therefore\ \frac{BD}{DC}=\frac{BA}{AC}.$$

NOTE. If AB = AC, the external bisector of $\angle$BAC is parallel to BC.

## THEOREM 74 (Second Proof)

**If a straight line through the vertex of a triangle divides the base internally or externally in the ratio of the other sides, it bisects the vertical angle internally or externally.**

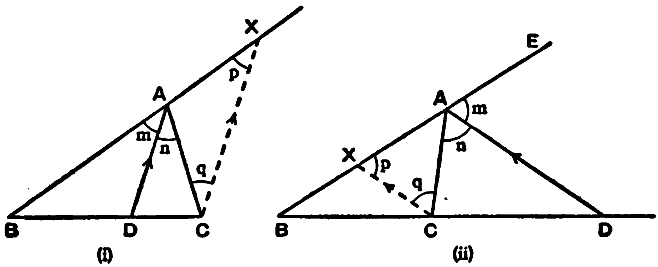



Fig. 822

**Given** a triangle ABC and a point D on BC, fig. 822 (i), or on BC produced, fig. 822 (ii), such that

$$\frac{BD}{DC}=\frac{AB}{AC}.$$

**To prove that** AD bisects $\angle$ BAC internally or externally.

**Construction.** Through C draw a line parallel to DA to cut BA, produced if necessary, at X.

**Proof.** With the notation in the figures,
since DA is parallel to CX,

$$\frac{BD}{DC}=\frac{BA}{AX}.$$

But $\frac{BD}{DC}=\frac{AB}{AC}$ *given*,

$\therefore$ AX = AC,

$\therefore$ $p=q$ *base $\angle$s, isos. $\triangle$.*

But $m=p$ *corr. $\angle$s*, DA $\parallel$ CX,

and $n=q$ *alt. $\angle$s*, DA $\parallel$ CX,

$\therefore$ $m=n$.

$\therefore$ AD bisects $\angle$ BAC, either internally or externally.

## EXERCISE 87

Nos. 1–5 refer to fig. 823 in which AD, AE are the bisectors of $\angle$BAC and cut BC at D, E.

1. If the line through D parallel to BA cuts CA at N, prove that $\frac{BA}{AC}=\frac{AN}{CN}$.

2. Prove that BD : DC = BE : CE.

[3] If the line through C parallel to AB cuts AD produced at K, prove that $\frac{AD}{DK}=\frac{AB}{AC}$.

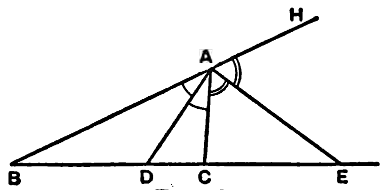


Fig. 823

4. If the line through E parallel to AB cuts AC produced at P, prove that PA : PC = AB : AC.

[5] If circles with B, C as centres are drawn through D and cut BA, CA at G, K, prove that GK is parallel to BC.

6. AX is a median of $\triangle$ABC; the bisectors of $\angle$AXB, $\angle$AXC meet AB, AC at H, K. Prove that HK is parallel to BC.

[7] A straight line cuts four lines OP, OQ, OR, OS at P, Q, R, S. If $\angle$POR = 90° and if OR bisects $\angle$QOS, prove that PQ : PS = QR : RS.

8. H is any point inside $\triangle$ABC; the bisectors of $\angle$BHC, $\angle$CHA, $\angle$AHB cut BC, CA, AB respectively at X, Y, Z. Prove that $\frac{BX}{XC}\times\frac{CY}{YA}\times\frac{AZ}{ZB}=1$.

[9] ABCD is a parallelogram. If the bisector of $\angle$BAD meets BD at X and CD at Y, prove that AX : XY = DC : DA.

10. ABCD is a quadrilateral in which AB = AD. The bisectors of $\angle$CAB, $\angle$CAD meet CB, CD respectively at H, K. Prove that HK is parallel to BD.

11. Two circles, centres A, B, touch at O. Any line parallel to AB cuts the circles at P, Q respectively. If AP and BQ are produced to meet at K, prove that KO is one of the bisectors of $\angle$AKB.

[12] ABCD is a quadrilateral such that $\angle B = \angle C$, and AC bisects $\angle BAD$. If BA and CD, when produced, meet at E, prove that $AD : DC = AE : BE$.

13. D is the mid-point of the base BC of $\triangle ABC$; AD is produced to E. If the internal bisectors of $\angle BDE$, $\angle CDE$ meet AB produced, AC produced at H, K respectively, prove that HK is parallel to BC.

[14] The diagonals AC, BD of the cyclic quadrilateral ABCD cut at E. If $AB = BC$, prove that $DA : DC = AE : EC$.

15. AB is a chord of a circle perpendicular to a diameter CD; E is any point on AB; CE, DE when produced meet the circle again at P, Q respectively. Prove that $PA : PB = QA : QB$.

16. AB, AC are equal chords of a circle and $\angle BAC > 60°$. If the tangent at C meets BA produced at T, prove that $CB : CA = TC : TA$.

[17] Prove the result in No. 16 if $\angle BAC < 60°$. [In this case, the tangent at C meets AB produced at T.]

18. **Apollonius' Circle.** A, B are fixed points and P is a variable point such that $PA : PB$ is constant.

Prove that

(i) the internal and external bisectors of $\angle APB$ cut AB at fixed points,

(ii) the locus of P is a circle.

If $PA : PB = 3 : 2$ and if $AB = 6$ cm., calculate the diameter of this circle.

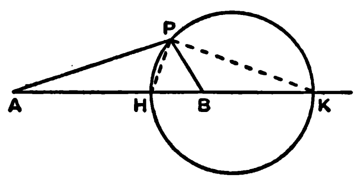



FIG. 824

[19] The tangent at a point A of a circle, centre O, meets a radius OB at T; AD is the perpendicular from A to OB. Prove that $DB : BT = AD : AT$.

*20. ABCD is a quadrilateral. If the bisectors of $\angle DAB$, $\angle DCB$ meet on DB, prove that the bisectors of $\angle ABC$, $\angle ADC$ meet on AC.

*21. Two circles touch internally at O; a chord PQ of the larger circle touches the smaller at R. Prove that $OP : OQ = PR : RQ$.

*22. ABCD is a parallelogram; the bisector of $\angle BAD$ meets BD at K; the bisector of $\angle ABC$ meets AC at L. Prove that LK is parallel to AB.

*23. In $\triangle ABC$, $\angle BAC = 90°$ and AD is an altitude. If the bisector of $\angle ABC$ meets AD, AC at L, K, prove that $AL : LD = CK : KA$.

## CONSTRUCTION 19

**Divide a given finite straight line in a given ratio, (i) internally, (ii) externally.**

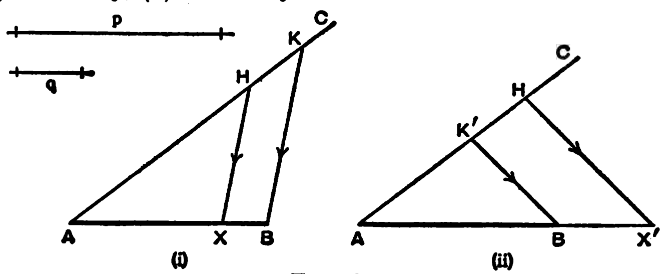



FIG. 825

**Given** two lines of lengths $p, q$ units and a finite line AB.

**To construct** (i) a point X in AB such that $\frac{AX}{XB} = \frac{p}{q}$;

(ii) a point X′ in AB produced such that $\frac{AX'}{BX'} = \frac{p}{q}$.

(i) **Construction.** Draw any line AC and cut off from it AH $= p$ units. HK $= q$ units. Join KB.

Through H draw HX parallel to KB to cut AB at X.

Then AB is divided internally at X in the ratio $p : q$.

**Proof.** Since XH is parallel to BK,

$$\frac{AX}{XB} = \frac{AH}{HK} = \frac{p}{q}.$$

(ii) **Construction.** Draw any line AC and cut off from it AH $= p$ units. From HA cut off HK′ $= q$ units. Join K′B.

Through H draw HX′ parallel to K′B to meet AB produced at X′.

Then AB is divided externally at X′ in the ratio $p : q$.

**Proof.** Since X′H is parallel to BK′,

$$\frac{AX'}{BX'} = \frac{AH}{K'H} = \frac{p}{q}.$$

## CONSTRUCTION 20

**Construct a fourth proportional to three given lines.**

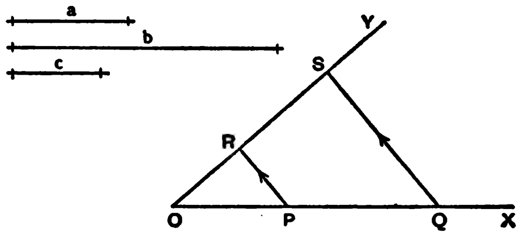



FIG. 826

**Given** three lines of lengths $a$, $b$, $c$ units.

**To construct** a line of length $d$ units such that $\frac{a}{b}=\frac{c}{d}$.

**Construction.** Draw any two lines OX, OY.

From OX cut off OP, OQ such that OP $=a$ units, OQ $=b$ units.

From OY cut off OR such that OR $=c$ units.

Join PR.

Through Q draw QS parallel to PR to meet OY at S.

Then OS is a fourth proportional to $a$, $b$, $c$.

**Proof.** Since PR is parallel to QS,

$$\frac{OP}{OQ}=\frac{OR}{OS},$$

$$\therefore \frac{a}{b}=\frac{c}{OS}.$$

NOTE. Constructing a third proportional to two given lines of lengths $a$, $b$ units is the same as constructing a fourth proportional to three lines of lengths $a$, $b$, $b$ units. Therefore the method of Construction 20 gives the required result.

## EXERCISE 88

1. Draw a line AB and divide it internally in the ratio 2 : 3.

2. Draw a line AB and divide it externally (i) in the ratio 5 : 3, (ii) in the ratio 3 : 5.

[3] Draw a line AB and divide it internally and externally in the ratio 4 : 7.

[4] Given a ruler graduated in cm. and mm., construct a line of length $1\frac{4}{7}$ cm.

5. Draw a line AB and construct points P, Q on it such that AP : PQ : QB = 2 : 6 : 3.

[6] Draw a line AB and construct a point X on AB and a point Y on AB produced such that AX : XB : BY = 4 : 5 : 2.

7. Construct and measure a fourth proportional to lines of lengths 4, 5, 6 cm.

[8] Construct and measure a third proportional to lines of lengths 5, 6 cm.

9. Use a construction to solve $\frac{x}{3}=\frac{7}{5}$.

10. Use a construction to find $x$ and $y$ such that $\frac{x}{3}=\frac{y}{4}$ and $x+y=11$.

Find graphically the values of the following:—

11. $\frac{2 \cdot 3 \times 5 \cdot 9}{4 \cdot 7}$. [12] $\frac{3 \cdot 7^2}{5 \cdot 2}$. 13. $3 \cdot 8 \times 2 \cdot 7$.

[14] Draw any triangle ABC and a line PQ. Construct a triangle such that its perimeter equals PQ and its sides are in the ratios BC : CA : AB.

15. Draw an angle BAC and take a point O on the bisector of $\angle$BAC, construct a line POQ cutting AB, AC at P, Q so that PO : OQ = 3 : 5.

16. Draw an equilateral triangle ABC, side 6 cm., and take a point N on BC so that BN = 1·5 cm. Construct a rectangle equal in area to $\triangle$ABC and such that one side is equal to AN. Measure an adjacent side.

17. Draw a line AB 3 inches long and construct a point P on AB so that $AP^2 = 2PB^2$.

[18] Draw a line AB 3 cm. long and construct a point Q on AB produced so that $AQ^2 = 3QB^2$.

## Similarity

The meaning of similarity has already been discussed, see pp. 70–72, where it was shown that to each test for the congruence of two triangles there corresponds a test for similarity. The tests for similarity were obtained by considering what groups of measurements of the sides and angles of a given triangle must be taken in order to draw elsewhere a triangle of the same shape but of any convenient size, *i.e.* in order to make a scale-drawing of the given triangle. Proofs that these tests are correct are given in Theorems 75–77.

**Definitions.** (1) Two polygons are said to be **equiangular to one another** if the angles of the first polygon, *taken in order*, are respectively equal to the angles of the second polygon taken in order.

(2) Two polygons are said to be **similar to one another** if (i) the polygons are equiangular to one another and if also (ii) the ratio of any side of the first polygon to the corresponding side of the second is the same, or can be proved to be the same, for every pair of corresponding sides, *i.e.* if corresponding sides are proportional.

It is obvious that polygons which are equiangular to one another need not be similar; consider, for example, a square and any rectangle. It is also obvious that polygons in which corresponding sides are proportional need not be similar; consider, for example, a square and any rhombus.

But the test for similarity of two triangles show that

(i) *triangles* equiangular to one another must be similar;
and (ii) *triangles* in which corresponding sides are proportional must be similar.

**Notation.** Similar triangles and polygons should always be named so that the order of the letters indicates the correspondence between the two figures.

Thus, the statement that $\triangle$s $\begin{matrix}ABC\\XYZ\end{matrix}$ are similar means that

$$\angle A = \angle X, \qquad \angle B = \angle Y, \qquad \angle C = \angle Z$$

and that

$$\frac{BC}{YZ} = \frac{CA}{ZX} = \frac{AB}{XY}.$$

## Examples for Oral Discussion

1. In fig. 827, if $\angle A = \angle X$, $\angle B = \angle Y$, $\angle C = \angle Z$, prove that

$$\frac{AB}{XY} = \frac{AC}{XZ} \quad \text{and that} \quad \frac{AB}{XY} = \frac{BC}{YZ}.$$

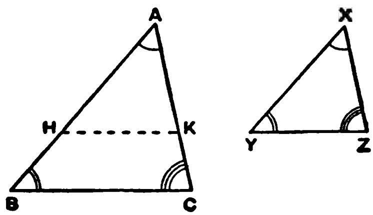



FIG. 827

From AB, AC cut off $AH = XY$, $AK = XZ$; join HK.

(i) Explain why $\angle AHK = \angle XYZ$.

(ii) Explain why $HK \parallel BC$. What follows?

(iii) What construction is needed to prove $\dfrac{AB}{XY} = \dfrac{BC}{YZ}$?

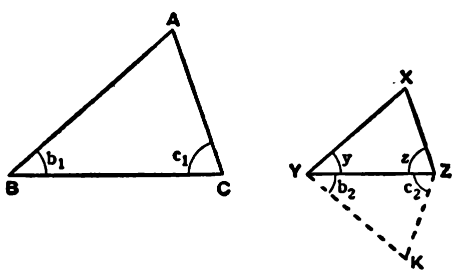



FIG. 828

2. In fig. 828, if $\dfrac{AB}{XY} = \dfrac{BC}{YZ} = \dfrac{CA}{ZX}$, prove that

$$\angle A = \angle X, \ \angle B = \angle Y, \ \angle C = \angle Z.$$

With the notation in fig. 828, take a point K such that $b_2 = b_1$ and $c_2 = c_1$.

(i) Using the result proved in No. 1, write down two ratios equal to $\dfrac{BC}{YZ}$. What follows from the data?

(ii) Prove that $\triangle$s $\begin{matrix} XYZ \\ KYZ \end{matrix}$ are congruent.

3. In fig. 829, if $\angle A = \angle X$ and if $\dfrac{AB}{XY} = \dfrac{AC}{XZ}$, prove that $\triangle$s $\begin{matrix} ABC \\ XYZ \end{matrix}$ are similar.

From AB, AC cut off

AH = XY, AK = XZ;

join HK.

(i) Explain why HK $\parallel$ BC.

(ii) Prove that $\triangle$s $\begin{matrix} AHK \\ XYZ \end{matrix}$ are congruent.

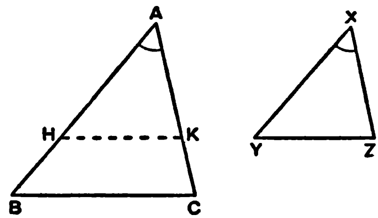


FIG. 829

4. In fig. 830, the triangles are *not* drawn accurately. The data are shown in the figure.

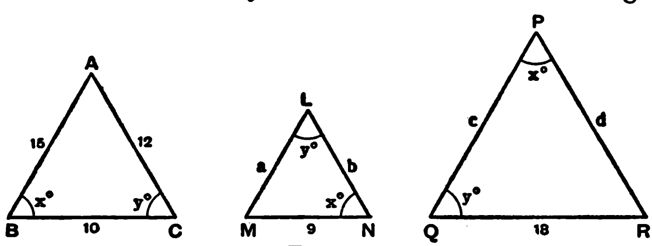


FIG. 830

(i) Explain why the triangles are similar and state the fact in the form $\triangle$s $\begin{matrix} ABC \\ \ldots \\ \ldots \end{matrix}$ are similar.

(ii) Find the values of *a*, *b*, *c*, *d*.

(iii) *Sketch* a triangle DEF such that EF = 6 cm., FD = 4·8 cm., DE = 4 cm. Is this triangle similar to $\triangle$ABC? If so, give the reason and state the fact in the proper form.

5. What is the simplest way of proving that the triangle whose sides are 5·1 in., 6·8 in., 8·5 in. is right-angled?

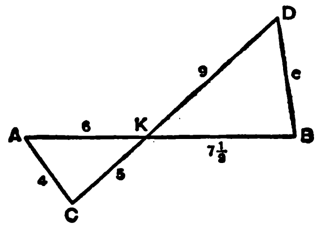



FIG. 831

6. In fig. 831, AKB and CKD are straight lines.

(i) Explain why the triangles are similar and state the fact in the proper form.

(ii) Find the value of *e*.

7. If $\Delta$s $\begin{matrix} ADK \\ QXB \end{matrix}$ are similar, name one ratio equal to (i) $\frac{AD}{AK}$, (ii) $\frac{BQ}{BX}$. Name two ratios equal to $\frac{QB}{AK}$.

8. Draw any triangle ABC and mark points D, E on AB, AC respectively so that $\angle ADE = \angle C$. Complete the sentence, $\Delta$s $\begin{matrix} ABC \\ \cdot\ \cdot\ \cdot \end{matrix}$ are similar. Name one ratio equal to AD : AE and two ratios equal to DE : CB.

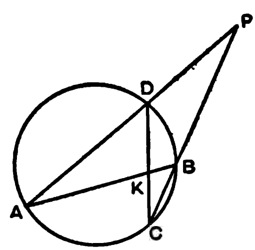



FIG. 832

9. In fig. 832, the chords AB, CD cut at K and the chords AD, CB are produced to cut at P. Name in the proper form, *with reasons*, two pairs of similar triangles.

Name one ratio equal to $\frac{KA}{KD}$ and two ratios equal to $\frac{PA}{PC}$.

## NUMERICAL EXAMPLES

### EXERCISE 89

[*Arrows indicate that lines are given parallel.*]

Nos. 1–4 refer to fig. 833, dimensions in inches.

1. If BH = 3″, find BK, HK.

[2] If BK = 4″, find BH, HK.

3. If HK = 4″, find BH, BK.

[4] If KC = 4″, find AH, HK.

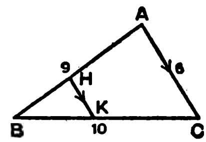



FIG. 833

5. Find the value of $p : q$ in fig. 834.

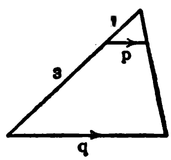



FIG. 834

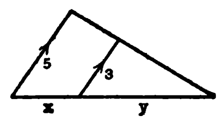



FIG. 835

6. Find the value of $y : x$ in fig. 835.

Find the marked lengths in Nos. 7–10, unit 1 cm.:

[7]

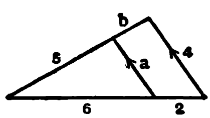



FIG. 836

8.

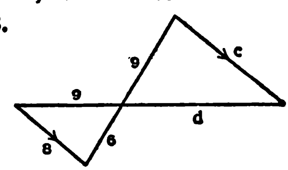



FIG. 837

[9]

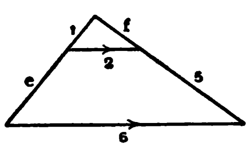



FIG. 838

10.

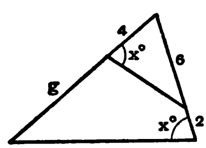



FIG. 839

11. A pole 10 ft. high casts a shadow $3\frac{1}{2}$ ft. long and at the same time a tower casts a shadow 42 ft. long, on level ground. Find the height of the tower.

[12] In a photograph of a chest of drawers, the height measures 6 in. and the breadth 3·2 in. If the height is $7\frac{1}{2}$ ft., find the breadth.

13. A light is 9 ft. above the floor; a ruler 8 in. long is held horizontally 4 ft. above the floor. Find the length of its shadow.

[14] The slope of a railway is marked as 1 in 60. What height in feet does it climb in $\frac{3}{4}$ mile?

15. In the quadrilateral ABCD, AB is parallel to DC; AB = 8 cm., AD = 3 cm., DC = 5 cm. If AD, BC produced meet at P, find PD.

[16] Show that the line joining the points (1, 1), (4, 2) is parallel to and half the line joining the points (0, 0), (6, 2).

[17] The bases of two equiangular triangles are 4 in., 6 in., and are corresponding sides. If the height of the first triangle is 5 in., find the area of the second.

18. A line parallel to BC meets AB, AC at X, Y; BC = 8 in., XY = 5 in.; the lines BC, XY are 2 in. apart. Find the area of $\triangle$ AXY.

19. The diameter of the base of a cone is 9 in. and its height is 15 in. Find the diameter of a section parallel to the base and 3 in. from it.

[20] The diameter of the base of a cone is 8 in.; the diameter of a parallel section, 3 in. from the base, is 6 in. Find the height of the cone.

Find the marked lengths in Nos. 21–24, unit 1 cm.:

[21]

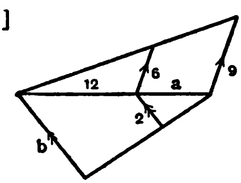



Fig. 840

22.

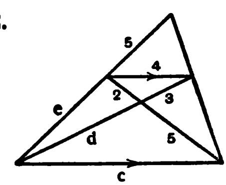



Fig. 841

23.

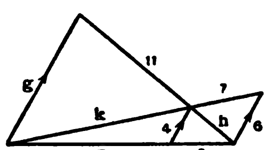



Fig. 842

[24]

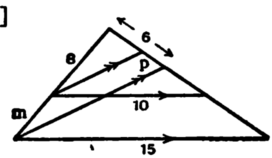



Fig. 843

[25] In $\triangle ABC$, $\angle B$ is a right angle; P is a point on AC; PQ, PR are the perpendiculars from P to BC, BA.

(i) If AB = 7 in., PQ = 1 in., PR = 2 in., find BC.

(ii) If AB = 7 in., BC = 5 in., PR = $x$ in., PQ = $y$ in., find an equation connecting $x$, $y$.

26. Explain why there are two similar triangles in fig. 844 (i) and in fig. 844 (ii). Name them in the proper way. What angle is equal to $\angle KCA$? What can you say about the four points A, B, C, D? What is the length of BD, unit 1 in.?

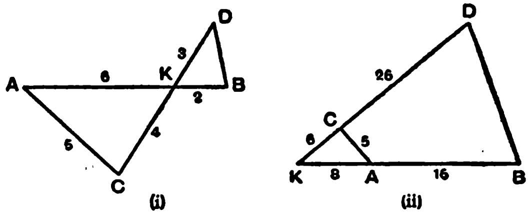



FIG. 844

27. If in fig. 845, AP : PB = 2 : 1 and if AC = 2BD, prove that PR = RQ.

28. If in fig. 845, AP : PB = 2 : 1 and if AC = 8 cm., BD = 5 cm., find PQ.

[29] A, B are points on the same side of a line OX and at distances 1 in., 5 in. from it; Q and R divide AB internally and externally in the ratio 5 : 3. Find the distances of Q and R from OX.

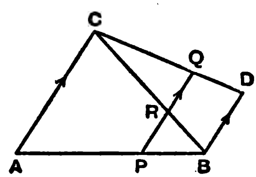



FIG. 845

[30] Three lines APB, AQC, ARD are cut by two parallel lines PQR, BCD. If AR = 3 in., RD = 2 in., BC = 4 in., find PQ.

Nos. 31–34 refer to fig. 846 in which the chords AC, BD cut at K, and the chords AD, BC are produced to meet at P.

31. If AK = 3″, KB = 2″, AB = 4″, DC = $1\frac{1}{2}$″, find KC, KD.

[32] If AK = 5″, BK = 4″, AC = 7″, find BD.

33. If PA = 9″, PB = 8″, AB = 4″, PC = 3″, find PD, CD.

[34] If PA = 9″, PB = 8″, AC = 6″, PC = 4″, find BD, PD.

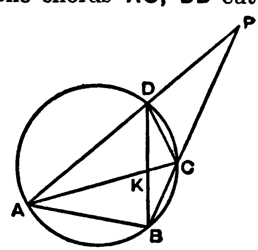



FIG. 846

35. In fig. 847, if $AB = 6$ cm., $AQ = 20$ cm., $AE = 5$ cm., $CD = 9$ cm., find $QD$ and $AP$.

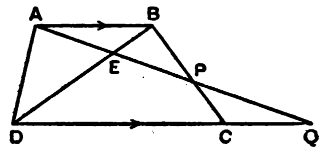


FIG. 847

36. $ABCD$ is a quadrilateral in which $\angle ABC = 90° = \angle ACD$, $AC = 5$ in., $BC = 3$ in., $CD = 10$ in. Find the distances of $D$ from $BC$ and $BA$.

37. How far in front of a pinhole camera must a man 6 ft. high stand in order that a full-length photograph may be taken on a film $2\frac{1}{4}$ in. high, $2\frac{1}{2}$ in. from the pinhole?

[38] A halfpenny (diameter 1 in.) at the distance of 3 yards appears nearly the same size as the sun or moon at its mean distance. Taking the distance of the sun as 93 million miles, find its diameter. Taking the diameter of the moon as 2160 miles, find its mean distance.

39. A rectangular table, 5 ft. wide, 8 ft. long, 3 ft. high, stands on a level floor under a hanging lamp. The shadow on the floor of the shorter side is 8 ft. long. Find the length of the shadow of the longer side and the height of the lamp above the table.

40. (i) A sphere, of radius 5 in., is placed inside a conical funnel whose *slant* height is 15 in. and whose greatest diameter is 18 in. Find the distance of the vertex of the funnel from the centre of the sphere.

(ii) Find the radius of a sphere which, when put in this funnel, touches the plane of the rim of the funnel.

[41] The length of each arm of a pair of nutcrackers is 6 in. Find the distance between the ends of the arms when a nut, diameter 1 in., is put with its nearer end 1 in. from the apex.

42. Draw a circle of radius 5 cm. and inscribe a triangle $ABC$ in the circle such that $BC : CA : AB = 5 : 6 : 7$.

*43. In $\triangle ABC$, $\angle B = 90°$, $AB = 5$ in., $BC = 2$ in. If the perpendicular bisector of $AC$ cuts $AB$ at $Q$, find $AQ$.

*44. A rectangular sheet of paper $ABCD$ is folded so that $D$ falls on $B$; the crease cuts $AB$ at $Q$. If $AB = 11$ in., $AD = 7$ in., find $AQ$.

*45. $PQ$ is a chord of a circle 5 cm. long; the tangents at $P$, $Q$ meet at $T$; $PR$ is a chord parallel to $TQ$. If $PT = 8$ cm., find $PR$.

*46. A, B, C are 3 points on level ground (see fig. 848); $AB = 9$ ft.; AP, BQ are vertical poles, each 8 ft. high; CR is a vertical post 5 ft. high. Straight lines PR, QR run from P, Q to R and are continued to meet the ground at Y, Z. Find the length of YZ.

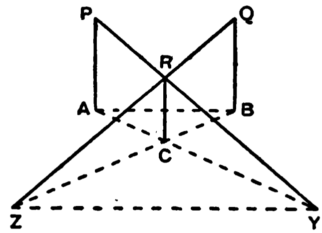


FIG. 848

*47. A cuboid, 2 in. by 3 in. by 4 in., rests with one of its largest faces on a table. A hollow cone, whose height is equal to its base-diameter, rests on the table covering the cuboid and touching its four upper corners. Find the height of the cone.

*48. Fig. 849 represents an object HK and its image PQ in a concave mirror, centre O, focus F; $CH = u$, $CP = v$, $CF = FO = f$, $HK = x$, $PQ = y$. Prove that (i) $\frac{1}{f} = \frac{1}{u} - \frac{1}{v}$; (ii) $y = \frac{vx}{u}$.

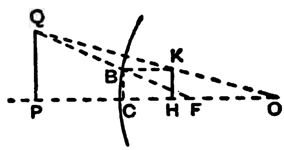


FIG. 849

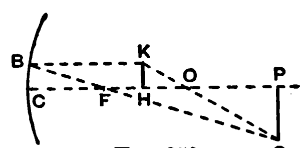


FIG. 850

*49. Fig. 850 also represents an object HK and its image PQ in a concave mirror. What do you notice about the image? With the notation and data of No. 48, prove that $\frac{1}{f} = \frac{1}{u} + \frac{1}{v}$.

*50. Fig. 851 represents an object HK and its image PQ in a thin concave lens, centre O, focus F; $OH = u$, $OP = v$, $OF = f$, $HK = x$, $PQ = y$. Prove that (i) $\frac{1}{f} = \frac{1}{v} - \frac{1}{u}$; (ii) $y = \frac{vx}{u}$.

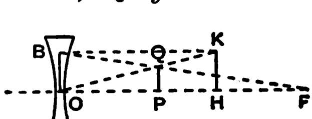


FIG. 851

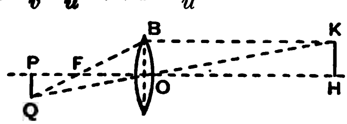


FIG. 852

*51. Fig. 852 represents an object HK and its image PQ in a thin convex lens. With the notation of No. 50, prove that $\frac{1}{f} = \frac{1}{u} + \frac{1}{v}$ and find $y$ in terms of $x$, $u$, $f$.

## THEOREM 75

**If two triangles are equiangular, their corresponding sides are proportional.**

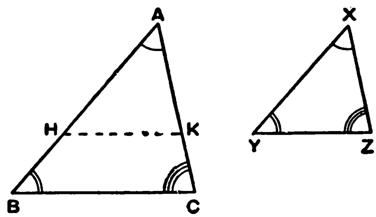



FIG. 853

**Given** two triangles ABC, XYZ in which

$$\angle A = \angle X, \quad \angle B = \angle Y, \quad \angle C = \angle Z.$$

**To prove that** $\dfrac{AB}{XY} = \dfrac{AC}{XZ} = \dfrac{BC}{YZ}$.

**Construction.** From AB, AC cut off AH, AK equal to XY, XZ.
Join HK.

**Proof.** In the $\triangle$s AHK, XYZ,

$AH = XY$ *constr.*,
$AK = XZ$ *constr.*,
$\angle A = \angle X$ *given*,

$\therefore$ $\triangle$s $\begin{matrix} AHK \\ XYZ \end{matrix}$ are congruent SAS.

$\therefore \angle AHK = \angle XYZ$,
but $\angle ABC = \angle XYZ$ *given*,
$\therefore \angle AHK = \angle ABC$.

But these are corresponding angles,

$\therefore$ HK is parallel to BC;

$\therefore$ HK divides AB, AC proportionally,

$$\therefore \frac{AB}{AH} = \frac{AC}{AK}.$$

But $AH = XY,\ AK = XZ$ *constr.*,

$$\therefore \frac{AB}{XY} = \frac{AC}{XZ}.$$

Similarly, by cutting off lengths from BA, BC equal to YX, YZ, it can be proved that

$$\frac{BA}{YX} = \frac{BC}{YZ}.$$

$$\therefore \frac{AB}{XY} = \frac{AC}{XZ} = \frac{BC}{YZ}.$$

*Abbreviation for reference:* equiangular $\triangle$s.

Theorem 75 is sometimes stated in the form,

**Equiangular triangles are similar.**

It has already been pointed out, see p. 481, that quadrilaterals which are equiangular need not be similar. See also p. 532, Paper 84, No. 4 (ii), and p. 533, Paper 86, No. 4 (i).

Attention should be called to the part of the proof printed in thick type. An additional construction is needed for proving that $\frac{BC}{YZ} = \frac{BA}{YX}$. The fact that HK is parallel to BC does not give a value for the ratio $\frac{BC}{HK}$. To say that $\frac{BC}{HK} = \frac{AB}{AH}$ is equivalent to assuming what has to be proved in this theorem.

The application of Theorem 75 to the principle of the **Diagonal Scale** should be pointed out and provides a useful opportunity for revision, and it is suggested that some oral examples similar to those on p. 190 should be taken.

**Important Hint.** In rider work, when writing down the three equal ratios of pairs of sides of two similar triangles ABC, XYZ, it is better to take the ratio of each side of $\triangle ABC$ to the corresponding side of $\triangle XYZ$,

$$\textit{i.e.}\quad \frac{BC}{YZ} = \frac{CA}{ZX} = \frac{AB}{XY},$$

than to write $\frac{AB}{BC} = \frac{XY}{YZ}$ and $\frac{BC}{CA} = \frac{YZ}{ZX}$.

### THEOREM 76 (First Proof)

**If the three sides of one triangle are proportional to the three sides of a second triangle, then the triangles are equiangular.**

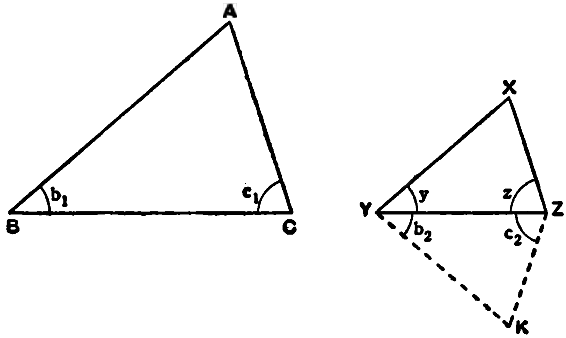



FIG. 854

**Given** two triangles ABC, XYZ in which

$$\frac{AB}{XY}=\frac{BC}{YZ}=\frac{CA}{ZX}.$$

**To prove that** $\angle A=\angle X,\quad \angle B=\angle Y,\quad \angle C=\angle Z.$

**Construction.** On the side of YZ opposite to X, draw YK, ZK so that

$$\angle KYZ=\angle B \text{ and } \angle KZY=\angle C.$$

**Proof.** With the notation in the figure,
in $\triangle$s ABC, KYZ,

$b_1=b_2$ *constr.*,
$c_1=c_2$ *constr.*,
$\therefore\ \angle A=\angle K$ *3rd $\angle$s of $\triangle$s.*

$\therefore\ \triangle\text{s}\ \begin{matrix}ABC\\KYZ\end{matrix}$ are similar *equiangular $\triangle$s.*

$$\therefore \frac{AB}{KY} = \frac{BC}{YZ}.$$

But $\frac{AB}{XY} = \frac{BC}{YZ}$ *given*,

$$\therefore \frac{AB}{KY} = \frac{AB}{XY}.$$

$$\therefore KY = XY.$$

Similarly, $KZ = XZ.$

$\therefore$ in $\triangle$s XYZ, KYZ,

$XY = KY$ *proved*,
$XZ = KZ$ *proved*,
$YZ = YZ.$

$\therefore$ $\triangle$s $\begin{matrix}XYZ\\KYZ\end{matrix}$ are congruent SSS.

$$\therefore y = b_2,$$

but $b_2 = b_1$ *constr.*,

$$\therefore y = b_1.$$

Similarly, $z = c_1.$

Therefore also $\angle X = \angle A$ *3rd $\angle$s of $\triangle$s.*

*Abbreviation for reference:* 3 sides proportional.

Theorem 76 is sometimes stated in the form,

**If the three sides of one triangle are proportional to the three sides of a second triangle, then the triangles are similar.**

## THEOREM 76 (Second Proof)

**If the three sides of one triangle are proportional to the three sides of a second triangle, then the triangles are equiangular.**

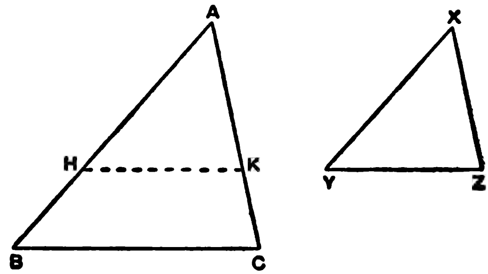



FIG. 855

**Given** two triangles ABC, XYZ in which

$$\frac{AB}{XY}=\frac{BC}{YZ}=\frac{CA}{ZX}.$$

**To prove that** $\angle A=\angle X$, $\angle B=\angle Y$, $\angle C=\angle Z$.

**Construction.** From AB, AC cut off AH, AK equal to XY, XZ.
Join HK.

**Proof.** $$\frac{AB}{XY}=\frac{AC}{XZ} \qquad \textit{given},$$

but $XY=AH$ and $XZ=AK$ *constr.*,

$$\therefore \frac{AB}{AH}=\frac{AC}{AK}.$$

$\therefore$ HK divides AB, AC proportionally,

$\therefore$ HK is parallel to BC,

$\therefore \angle ABC=\angle AHK$ and $\angle ACB=\angle AKH$ *corr.* $\angle$*s.*

$\therefore \triangle$s $\begin{matrix}ABC\\AHK\end{matrix}$ are equiangular,

$$\therefore \frac{BC}{HK}=\frac{AB}{AH} \qquad \textit{corr. sides proportional},$$

but $\qquad \mathsf{AH} = \mathsf{XY} \qquad$ *constr.*,

$$\therefore \frac{\mathsf{BC}}{\mathsf{HK}} = \frac{\mathsf{AB}}{\mathsf{XY}};$$

but $\qquad \dfrac{\mathsf{BC}}{\mathsf{YZ}} = \dfrac{\mathsf{AB}}{\mathsf{XY}} \qquad$ *given*,

$$\therefore \mathsf{HK} = \mathsf{YZ}.$$

$\therefore$ in $\triangle$s AHK, XYZ,

$\mathsf{AH} = \mathsf{XY} \qquad$ *constr.*,
$\mathsf{AK} = \mathsf{XZ} \qquad$ *constr.*,
$\mathsf{HK} = \mathsf{YZ} \qquad$ *proved*,

$\therefore$ $\triangle$s $\begin{matrix}\mathsf{AHK}\\ \mathsf{XYZ}\end{matrix}$ are congruent $\qquad$ SSS.

$\therefore \angle \mathsf{A} = \angle \mathsf{X}$ and $\angle \mathsf{AHK} = \angle \mathsf{Y}$ and $\angle \mathsf{AKH} = \angle \mathsf{Z}$,

but $\quad \angle \mathsf{AHK} = \angle \mathsf{B}$ and $\angle \mathsf{AKH} = \angle \mathsf{C} \qquad$ *proved*,

$\therefore \angle \mathsf{B} = \angle \mathsf{Y}$ and $\angle \mathsf{C} = \angle \mathsf{Z}$.

NOTE. The advantage of the second proof of Theorem 76 lies in the fact that it depends on the same construction as is used for Theorems 75, 77; the proof itself is, however, rather more difficult.

**Hints for Rider Work.** In many riders it is required to prove that $\frac{p}{q} = \frac{r}{s}$, where $p$, $q$, $r$, $s$ are the lengths of lines in a given figure.

(i) To prove that $\frac{p}{q} = \frac{r}{s}$ is equivalent to proving that $\frac{p}{r} = \frac{q}{s}$.

(ii) See whether there are two triangles one of which has $p$, $q$ as sides and the other $r$, $s$ as sides, or one of which has $p$, $r$ as sides and the other $q$, $s$ as sides, and try to prove these triangles are similar.

(iii) See whether the ratio $\frac{p}{q}$, or the ratio $\frac{p}{r}$, can be replaced by a more convenient equal ratio by using parallels or similar triangles.

## THEOREM 77

**If two triangles have an angle of one equal to an angle of the other, and the sides about these equal angles proportional, the triangles are equiangular.**

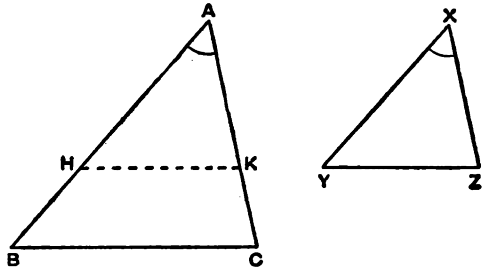



FIG. 856

**Given** two triangles ABC, XYZ in which

$$\angle A = \angle X \quad \text{and} \quad \frac{AB}{XY} = \frac{AC}{XZ}.$$

**To prove that** $\angle B = \angle Y$ and $\angle C = \angle Z$.

**Construction.** From AB, AC cut off AH, AK equal to XY, XZ. Join HK.

**Proof.** In $\triangle$s AHK, XYZ,

$$AH = XY \quad \textit{constr.},$$
$$AK = XZ \quad \textit{constr.},$$
$$\angle A = \angle X \quad \textit{given},$$

$$\therefore \triangle\text{s} \begin{matrix} AHK \\ XYZ \end{matrix} \text{ are congruent} \qquad SAS.$$

$$\therefore \angle AHK = \angle Y \quad \text{and} \quad \angle AKH = \angle Z.$$

Also $\dfrac{AB}{XY} = \dfrac{AC}{XZ}$ *given,*

and $XY = AH, \quad XZ = AK$ *constr.*

$$\therefore \frac{AB}{AH} = \frac{AC}{AK},$$

$\therefore$ HK divides the sides AB, AC proportionally,

$\therefore$ HK is parallel to BC.

$\therefore$ $\angle AHK = \angle B$ and $\angle AKH = \angle C$ *corr.* $\angle$*s.*

But $\angle AHK = \angle Y$ and $\angle AKH = \angle Z$ *proved,*

$\therefore$ $\angle B = \angle Y$ and $\angle C = \angle Z$.

*Abbreviation for reference:* ratio of 2 sides, inc. $\angle$.

## EXERCISE 90

[*Arrows indicate that lines are given parallel.*]

Nos. 1–4 refer to fig. 857. Name two ratios equal to the ratios in Nos. 1–4:

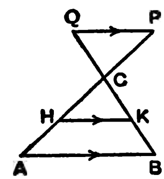


FIG. 857

1. $\dfrac{HK}{AB}$. [2] $\dfrac{AB}{PQ}$.

[3] CH : CK. 4. AB : BC.

Nos. 5–10 refer to fig. 858.

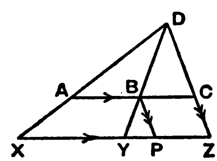


FIG. 858

5. Prove that $\dfrac{BC}{YZ} = \dfrac{DA}{DX}$.

[6] Prove that $\dfrac{XY}{YZ} = \dfrac{AB}{BC}$.

7. Prove that YP : AX = BC : AD.

[8] Prove that BP : DZ = XA : XD.

9. If the line joining C to Y is parallel to DX, prove that

BC : XY = YZ : XZ.

10. If XB is joined and produced to meet the line through D parallel to XZ at R, prove that BP : XB = DZ : XR.

[11] With the data of fig. 801, p. 461, prove that

QK : AH = PC : AB.

[12] With the data of fig. 812, p. 466, prove that

CQ : PA = CR : RQ.

13. BE, CF are altitudes of $\triangle$ABC. Prove that

BE : CF = AB : AC.

14. ABC is a triangle inscribed in a circle. The bisector of $\angle$BAC cuts BC at Q and cuts the circle again at P. Prove that AC : AP = AQ : AB and name a ratio equal to BQ : AB.

[15] AB is a diameter of a circle ABP; PT is the perpendicular from P to the tangent at A. Prove that PT : PA = AP : AB.

16. In $\triangle$ABC, $\angle$BAC = 90°; AD is an altitude. Prove that DC : AC = AC : BC and name two ratios equal to CD : DA.

[17] The medians BY, CZ of $\triangle$ABC intersect at G. Prove that GY = $\frac{1}{3}$BY. [Join YZ.]

Nos. 18–25 refer to fig. 859 in which XAD, AQZ, CQYX are straight lines.

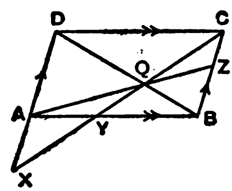



FIG. 859

18. Complete the relations:

$$\frac{QC}{QX} = \frac{BC}{\dots} = \frac{BY}{\dots} = \frac{QY}{\dots}.$$

[19] Complete the relations:

$$\frac{BZ}{AD} = \frac{BQ}{\dots} = \frac{BY}{\dots}.$$

20. Prove that CQ : QX = CY : CX.

[21] Prove that AD : BY = DX : AB.

22. If BZ = 2ZC, find the values of (i) $\frac{QD}{QB}$, (ii) $\frac{DQ}{DB}$.

[23] If DA = 3AX, find the values of (i) $\frac{BQ}{QD}$, (ii) $\frac{BZ}{ZC}$.

24. If AY = YB, prove that XQ = 2QC.

*25. Prove that $\triangle$CDX : $\triangle$BDA = $\triangle$BDA : $\triangle$BCY.

[26] ABCDE is a regular pentagon; AX, AY are the perpendiculars from A to CD, CB produced. Prove that AX : AY = AD : AB.

27. The bisector of $\angle$BAC meets BC at D; X is a point on AD, produced if necessary, such that CX = CD. Prove that $\triangle$s ABD, ACX are similar and deduce that AB : AC = BD : DC.

[28] M is the mid-point of AB; AXB, MYB are equilateral triangles on opposite sides of AB; XY cuts AB at Z. Prove that AZ = 2ZB.

29. BE, CF are altitudes of $\triangle$ABC. Prove that

EF : BC = AF : AC.

[30] Prove that the common tangents of two non-intersecting circles divide, internally and externally, the line joining the centres in the ratio of the radii.

31. If $\triangle$s $\begin{matrix} ABC \\ XYZ \end{matrix}$ are similar and if AP, XQ are medians, prove that $\angle BAP = \angle YXQ$. [Use Theorem 77.]

[32] ABCD is a cyclic quadrilateral and AK is the perpendicular from A to BD. If AC : AB = CD : KB, prove that AC is a diameter.

33. In fig. 860, $\angle ABK = \angle ACD$ and $\angle AKB = \angle ADC$. Prove that $\triangle$s ACB, ADK are similar. [Use Theorem 77.]

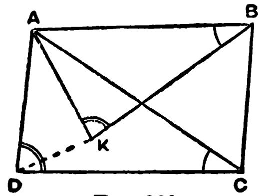


FIG. 860

[34] A radius CB of a circle, centre C, is produced to D so that CB = BD. If M is the mid-point of CB and if P is any point on the circle, prove that

$$\angle CPM = \angle CDP.$$

35. ABPQ, ABRS are two circles. If PAS, QAR are straight lines, prove that BP : BQ = BS : BR.

[36] ABCD is a rectangle. Two perpendicular lines are drawn; one cuts AB, CD at E, F; the other cuts AD, BC at G, H. Prove that EF : GH = BC : AB.

37. P is a variable point on a given circle, centre A; O is a fixed point outside the circle; Q is a point on OP such that $OQ = \frac{1}{3}OP$. Prove that the locus of Q is a circle. [Draw QK $\parallel$ PA to cut OA at K.]

*38. ABCD is a quadrilateral in which AC bisects $\angle BAD$ and $\angle ACD = \angle ABC$. If X, Y are the mid-points of BC, CD, prove that A, X, C, Y are concyclic. [Join AX, AY.]

*39. ABC is a triangle inscribed in a circle; the tangent at C cuts the line through B parallel to AC at D. Prove that $CD^2 : AB^2 = DB : CA$.

*40. ABCD is a quadrilateral in which AB is parallel to DC, and $\angle D = \angle C < 90°$. If P is a point on CD such that $\angle APB = \angle C$, prove that $DP : PC = PA^2 : PB^2$.

*41. AB, DC are the parallel sides of a trapezium ABCD; any line parallel to AB cuts CA, CB at H, K; DH, DK cut AB and AB produced at X, Y. Prove that AB = XY.

*42. In $\triangle ABC$, $\angle BAC = 90°$; ABXY, ACZW are squares outside $\triangle ABC$. If BZ, CX cut AC, AB at K, H, prove that AH = AK.

## Ratios and Areas

The distinction between the meanings of AB : XY and AB . XY needs emphasis.

The ratio AB : XY is represented by the *fraction* $\frac{AB}{XY}$; but AB.XY denotes the area of a rectangle whose adjacent sides are equal to AB and XY and is measured by the *product* of the numbers of units of length in AB and XY.

If $$\frac{c}{d}=\frac{p}{q}, \quad \text{then} \quad cq=dp;$$

and if $$\frac{AB}{XY}=\frac{CD}{ZW}, \quad \text{then} \quad AB.ZW=CD.XY.$$

Thus if AB : XY = CD : ZW, the rectangle contained by AB and ZW is equal in area to the rectangle contained by CD and XY.

Conversely, if the rectangles ABCD, APQR are equal in area, AB. AD = AP. AR;

and it then follows that $\frac{AB}{AP}=\frac{AR}{AD}$ and $\frac{AB}{AR}=\frac{AP}{AD}$.

The *geometrical* proof of this algebraic argument forms a useful exercise.

### Example for Oral Discussion

**In fig. 861, BAP, DAR are perpendicular lines such that the rectangles BADC, PARQ are equal in area.**

**Prove that**

AB : AP = AR : AD.

Join BR, DP, BD, PR.

(i) Prove that

$\triangle$BDR = $\triangle$BPR.

(ii) What can you now say about the lines BR, DP?

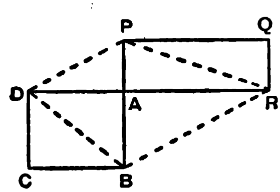


Fig. 861

**Hint on Rider Work.** If it is required to prove a rectangle-property, it is often best to convert it into a statement about equal ratios.

## EXERCISE 91

**1.** Express as equal products, (i) $x : a = y : b$; (ii) $p : q = q : r$.

**2.** Express as equal ratios, (i) $pq = rs$; (ii) $x^2 = yz$.

Draw a triangle ABC and a line parallel to BC cutting AB, AC at H, K respectively and prove the statements in Nos. 3–6:

**3.** AH . KC = AK . HB. [4] AH . AC = AK . AB.

**5.** AH . BC = AB . HK. [6] AC . HK = AK . BC.

**7.** Sketch a rectangle ABCD equal in area to a square EFGH. Using large letters represent the data (i) by two equal products, (ii) by two equal ratios.

**8.** What can you say about the rectangles ABCD, KLMN if (i) AB : KL = KN : BC; (ii) DA : NK = AB : KL?

**9.** With the data of fig. 857, p. 497, obtain products equal to (i) CA . CQ; (ii) CH . PQ.

[10] With the data of fig. 858, p. 497, prove that

DB . XZ = AC . DY.

**11.** P, Q are points on the sides AB, AC respectively of $\triangle$ ABC, such that $\angle$ APQ = $\angle$ ACB. Prove that AP . AB = AQ . AC.

[12] The altitudes BE, CF of $\triangle$ ABC intersect at H. Prove that BH . HE = CH . HF.

Nos. 13–14 refer to fig. 862 in which $\angle$BAC = 1 rt. $\angle$, and AD is perpendicular to BC.

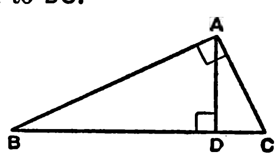



FIG 862

**13.** By using equal ratios, prove that $BA^2$ = BD . BC. What can you say about $CA^2$?

**14.** Prove that $AD^2$ = BD . DC.

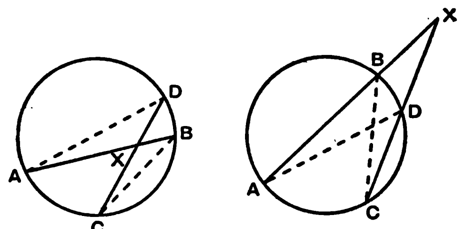
## Examples for Oral Discussion

1. Fig. 863 represents two chords AB, CD of a circle intersecting (when produced if necessary) at X. Prove that

$$XA.XB = XC.XD.$$

Join AD, BC. What ratios must you prove equal?

Explain why $\triangle$s $\begin{matrix}AXD\\CXB\end{matrix}$ are similar in each figure.




(i) FIG. 863 (ii)

2. In fig. 864, a chord AB of a circle ABT meets, when produced, at X the tangent to the circle at T. Prove that

$$XA.XB = XT^2.$$

Join TA, TB. What ratios must you prove equal?

Explain why $\triangle$s $\begin{matrix}AXT\\TXB\end{matrix}$ are similar.

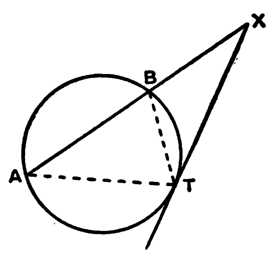



FIG. 864

3. In fig. 863 (i), if AX = 4 cm., XB = 6 cm., CX = 3 cm., find CD.

4. In fig. 863 (ii), if AB = 6 cm., BX = 4 cm., DX = 5 cm., find CD.

5. In fig. 864, if XB = 4 cm., XT = 6 cm., find AB.

**6. Fig. 865 represents two straight lines AB, CD which are divided both internally, or both externally, at the same point X. If XA.XB = XC.XD, prove that A, B, C, D are concyclic.**

Join AD, BC.

Express the data in the form of two equal ratios and then explain why $\triangle$s $\begin{matrix}AXD\\CXB\end{matrix}$ are similar in each figure.

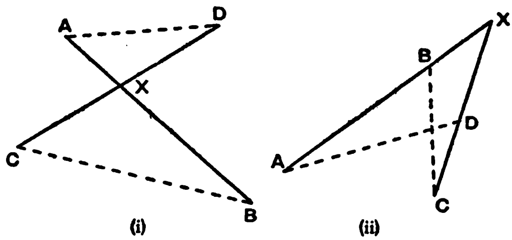



FIG. 865

## NUMERICAL EXAMPLES

### EXERCISE 92

[*For additional examples, see Exercise* 78, *p.* 423.]

1. If in fig. 863 (i), p. 502, AB = 9 cm., AX = 4 cm., CX = 2·5 cm., find CD.

2. If in fig. 863 (ii), p. 502, AB = 7 cm., BX = 3 cm., DX = 4 cm., AD = 9 cm., find CD and BC.

Nos. 3–6 refer to fig. 864, p. 502.

[3] If AB = 9 cm., BX = 3 cm., find XT.

4. If BX = 6 cm., TX = 12 cm., find AB.

[5] If AX = 3 in., AB = 2 in., AT = 4 in., find BT.

*6. If AB = 8 in., AT = 6 in., BT = 5 in., find XT.

7. PN is the perpendicular from a point P on a circle to a diameter AB. If AN = 5 cm., NB = 7·2 cm., find PN.

[8] In $\triangle$ABC, $\angle$BAC = 1 rt. $\angle$; AD is an altitude. If AB = 5 in. and AC = 12 in., find BD.

9. In $\triangle$ABC, AB = 8 cm., AC = 12 cm.; a circle through B, C cuts AB, AC at P, Q. If BP = 5 cm., find CQ.

10. ABC is a triangle inscribed in a circle; AB = AC = 10 in., BC = 12 in.; the perpendicular AD from A to BC is produced to meet the circle at E. Find DE and the radius of the circle.

## THEOREM 78

**If a perpendicular is drawn from the right angle of a right-angled triangle to the hypotenuse, the triangles on each side of the perpendicular are similar to the whole triangle and to one another.**

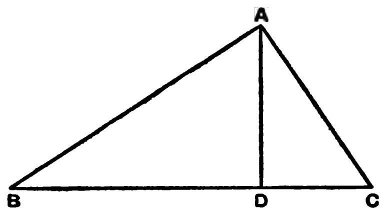



FIG. 866

**Given** a triangle ABC in which $\angle$A is a right angle, and the perpendicular AD from A to BC.

**To prove that** $\triangle$s ABC, DBA, DAC are similar.

**Proof.** In $\triangle$s ABC, DBA,

$\angle$BAC $=$ $\angle$BDA *rt.* $\angle$*s, given,*
$\angle$ABC $=$ $\angle$DBA *same angle,*

$\therefore$ the third angles ACB, DAB are equal.

$\therefore$ $\triangle$s $\begin{matrix}\text{ABC}\\ \text{DBA}\end{matrix}$ are equiangular.

In the same way it may be proved that

$\triangle$s $\begin{matrix}\text{ABC}\\ \text{DAC}\end{matrix}$ are equiangular.

But triangles which are equiangular are also similar.

$\therefore$ $\triangle$s ABC, DBA, DAC are similar.

**Corollary 1. The square on the perpendicular to the hypotenuse is equal to the rectangle contained by the segments of the hypotenuse, that is**

$$AD^2 = BD \, . \, DC.$$

Since $\triangle$s $\begin{matrix} DBA \\ DAC \end{matrix}$ are similar,

$$\frac{DA}{DB} = \frac{DC}{DA},$$

$$\therefore \; DA^2 = DB \, . \, DC.$$

**Corollary 2. The square on either of the sides containing the right angle is equal to the rectangle contained by the hypotenuse and the segment of the hypotenuse adjacent to that side, that is**

$$BA^2 = BD \, . \, BC \quad \text{and} \quad CA^2 = CD \, . \, CB.$$

Since $\triangle$s $\begin{matrix} ABC \\ DBA \end{matrix}$ are similar,

$$\frac{BA}{BD} = \frac{BC}{BA},$$

$$\therefore \; BA^2 = BD \, . \, BC.$$

Similarly, $\quad CA^2 = CD \, . \, CB.$

Alternative methods of proof of these Corollaries have been given on p. 429. The deduction of Corollary 2 from the ordinary proof of Pythagoras' theorem was given on p. 283.

On the other hand, the proof of Corollary 2 by means of similar triangles provides an alternative method for proving Pythagoras' theorem and is usually adopted in French text-books.

The results given in these Corollaries may also be stated as follows, see the definition on p. 454.

**If AD is an altitude of $\triangle$ ABC and if $\angle$ BAC is a right angle,**

**(i) AD is a mean proportional between BD and DC;**

**(ii) BA is a mean proportional between BD and BC;**
**CA is a mean proportional between CD and CB.**

## THEOREM 79

**If two chords of a circle, produced if necessary, cut one another, the rectangle contained by the segments of the one is equal to the rectangle contained by the segments of the other.**

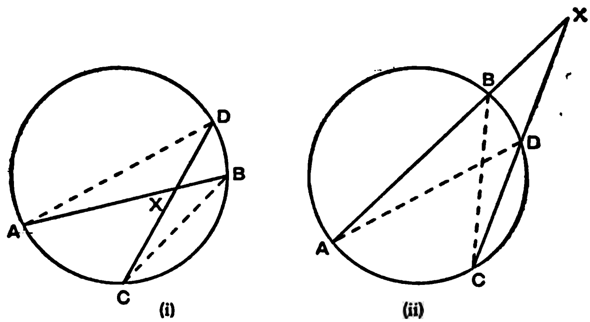



FIG. 867

**Given** two chords AB, CD of a circle intersecting at X, either inside, fig. 867 (i), or outside, fig. 867 (ii), the circle.

**To prove that** XA . XB = XC . XD.

**Construction.** Join BC, AD.

**Proof.** In $\triangle$s AXD, CXB,

$\angle$A = $\angle$C *same segment,*

$\angle$AXD = $\angle$CXB *vert. opp.* $\angle$*s*, fig. (i).
*same* $\angle$, fig. (ii),

$\therefore$ the third angles ADX, CBX are equal,

$\therefore$ $\triangle$s $\begin{matrix}\text{AXD}\\ \text{CXB}\end{matrix}$ are equiangular,

$\therefore$ $\dfrac{\text{XA}}{\text{XC}} = \dfrac{\text{XD}}{\text{XB}}$ *corr. sides proportional,*

$\therefore$ XA . XB = XC . XD.

*Abbreviation for reference:* intersecting chords.

## THEOREM 80

**If from any point outside a circle, a secant and a tangent are drawn, the rectangle contained by the whole secant and the part of it outside the circle is equal to the square on the tangent.**

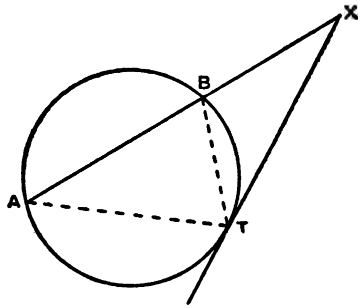



FIG. 868

**Given** the tangent XT to a circle from a point X outside the circle and a straight line XBA cutting the circle at B, A.

**To prove that** $XA . XB = XT^2$.

**Construction** Join TA, TB.

**Proof.** In the $\triangle$s AXT, TXB,

$\angle TAX = \angle BTX$ *alt. segment,*
$\angle AXT = \angle TXB$ *same $\angle$,*

$\therefore$ the third angles ATX, TBX are equal,

$\therefore$ $\triangle$s $\frac{AXT}{TXB}$ are equiangular,

$\therefore$ $\frac{XA}{XT} = \frac{XT}{XB}$ *corr. sides proportional,*

$\therefore$ $XA . XB = XT^2$.

*Abbreviation for reference:* tangent property.

## THEOREM 81

**If two straight lines AB and CD are divided both internally or both externally at the same point X such that**

$$XA \cdot XB = XC \cdot XD,$$

**the four points A, B, C, D are concyclic.**

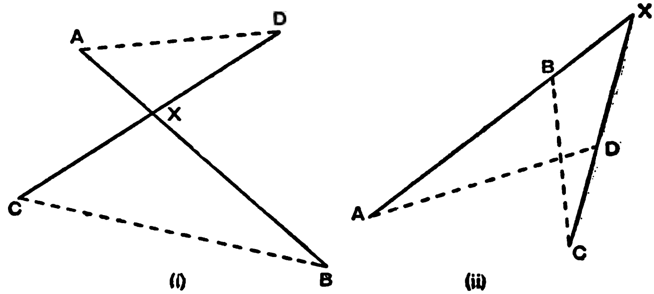



FIG. 869

**Construction.** Join AD, BC.

**Proof.** In $\triangle$s AXD, CXB,

$\angle AXD = \angle CXB$ *vert. opp.* $\angle$*s*, fig. (i), *same* $\angle$, fig. (ii),

and $\dfrac{XA}{XD} = \dfrac{XC}{XB}$ *because* $XA \cdot XB = XC \cdot XD$.

$\therefore$ $\triangle$s $\begin{matrix} AXD \\ CXB \end{matrix}$ are equiangular *ratio of 2 sides, inc.* $\angle$.

$\therefore$ $\angle DAX = \angle BCX$,

that is, $\angle DAB = \angle DCB$,

$\therefore$ DB subtends equal angles at points A, C on the same side of DB,

$\therefore$ D, B, A, C lie on a circle.

NOTE. In Theorems 81, 82, it is unnecessary to start by stating what is given and what is to be proved, because this is merely a repetition of the enunciation which is here given in terms of the letters to be used.

## THEOREM 82

**If the straight line AB is divided externally at X, and if C is a point, not on AB, such that**

$$XA . XB = XC^2,$$

**the circle ABC touches XC at C.**

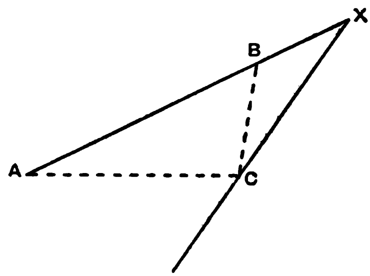



FIG. 870

**Construction.** Join CA, CB.

**Proof.** In $\triangle$s AXC, CXB,

$\angle AXC = \angle CXB$ *same* $\angle$,

and $\dfrac{XA}{XC} = \dfrac{XC}{XB}$ because $XA . XB = XC^2$.

$\therefore$ $\triangle$s $\begin{matrix} AXC \\ CXB \end{matrix}$ are equiangular *ratio of* 2 *sides, inc.* $\angle$,

$\therefore$ $\angle CAX = \angle BCX$.

$\therefore$ CX touches the circle CAB at C *conv. alt. segment.*

NOTE. Alternative proofs of Theorems 79–82 have been given in Theorems 67–69, pp. 426–428.

Theorem 80 may be deduced from Theorem 79 by taking the limiting case when D coincides with C in fig. 867 (ii), so that XDC becomes a tangent.

Similarly, Theorem 82 may be deduced from Theorem 81.

# NUMERICAL EXAMPLES

## EXERCISE 93

1. AXB, CXD are two perpendicular chords of a circle, centre O; AX = 6 cm., CX = 10 cm., DX = 12 cm.; find OX and OA.

2 The diagonals of the quadrilateral ABCD cut at K. If KA = 4 cm., KB = 6 cm., AC = 16 cm., BD = 14 cm., prove that ABCD is cyclic.

[3] H and K are points on the sides AB, AC respectively of $\triangle$ABC. If AH = 4 cm., HB = 11 cm., AK = 5 cm., KC = 7 cm., prove that HKCB is a cyclic quadrilateral.

4. In $\triangle$ABC, AB = 18 cm., BC = 12 cm., CA = 13·5 cm.; K is a point on AB such that AK = 10 cm. Prove that BC is a tangent to the circle AKC and find the length of CK.

[5] In $\triangle$ABC, $\angle$B = 90°, AB = 3 in., BC = 4 in.; the circle which touches BC at C and passes through A is drawn. If it cuts BA produced at P, find AP and the radius of the circle.

6. If in fig. 864, p. 502, XA = 2XT, prove that AB = 3BX.

7. If in fig. 864, p. 502, XT = $\frac{2}{3}$XA, find the ratio XB : BA.

*8. If in fig. 864, p. 502, BT = $p$ cm., TA = $q$ cm., AB = $r$ cm., BX = $x$ cm., TX = $y$ cm., find $x$ and $y$ in terms of $p$, $q$, $r$.

Prove also that $\dfrac{\mathrm{XA}}{\mathrm{XB}} = \dfrac{q^2}{p^2}$.

*9. X is the mid-point of a line TY of length 2 in.; TZ is drawn so that $\angle$ZTX = 45°. A circle is drawn through X and Y and to touch TZ at P. Prove that $\angle$TXP = 90°, and find the radius of the circle.

*10. (Spherometer.) The vertices A, B, C of an equilateral triangle lie on a sphere; the diameter PQR of the sphere perpendicular to the plane ABC cuts it at Q. If PQ = 1 in. and AB = 6 in., find the diameter of the sphere.

*11. A sphere of diameter 6 in. rests in a 5-in. diameter hole in a table. A thin square plate of side 10 in. with a central hole of diameter 2 in. rests with one edge AB on the table and the circumference of the hole lying on the sphere. Find the distance of AB from the centre of the 5-in. hole.

## Examples for Oral Discussion

**1. In fig. 871, AE is a diameter of the circumcircle of △ABC. If AD is an altitude of △ABC, prove that**

AB . AC = AD . AE.

Join BE.

(i) What ratios must you prove equal?

(ii) Prove that △s ABE, ADC are similar.

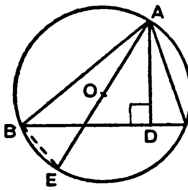


FIG. 871

**2. Ptolemy's Theorem. If ABCD is a cyclic quadrilateral, prove that**

BC . AD + AB . CD = AC . BD.

Take a point P on BD so that ∠DAP = ∠BAC.

(i) Prove that BC . AD = AC . DP.

(ii) Prove that AB . CD = AC . BP.

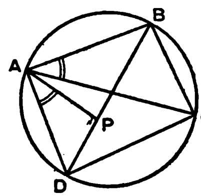


FIG. 872

3. What special result can be deduced from Ptolemy's theorem if, in fig. 872, △ABC is equilateral?

## EXERCISE 94

[*For additional examples, see Exercise* 79, p. 430.]

1. The diagonals of a cyclic quadrilateral ABCD intersect at K. Prove that AD . KC = BC . KD.

[2] Two lines XAB, XCD cut a circle at A, B, C, D. Prove that XA . BC = XC . AD.

3. If in fig. 874, p. 512, XB = BT, prove that XA . XB = TA$^2$

4. The sides BA, CA of △ABC are produced to D, E respectively so that ∠AED = ∠ABC. Prove that

AB . AD = AC . AE.

[5] In △ABC, ∠B = 90°; P is any point on AB. If the circle, diameter AP, cuts AC at Q, prove that AP . AB = AQ . AC.

6. In △ABC, ∠C = 90°; a circle is drawn to touch AC at C and cuts AB at P, Q. If CN is an altitude of △ABC, prove that AN : AP = AQ : AB.

7. PQ, RS are chords of the circles ABPQ, ABRS, which meet, when produced if necessary, at a point on AB or AB produced. Prove that P, Q, S, R are concyclic.

8. In fig. 873, AP, AQ are tangents to the circles ABQ, ABP. Prove that $AB^2 = BP.BQ$.

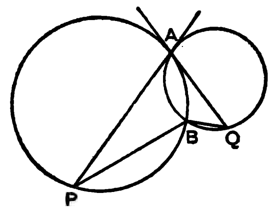


FIG. 873

[9] AC is a chord of a circle. From a point P on the minor arc AC, lines PD, PE are drawn parallel to the tangents at A, C cutting AC at D, E respectively. Prove that

(i) $\Delta$s APD, PCE are similar,
(ii) $AD.CE = PD^2$.

10. In $\triangle ABC$, $AB = AC$; D is a point on AC such that $BD = BC$. Prove that $BC^2 = AC.CD$.

[11] Two chords AB, CD of a circle intersect at X. If D is the mid-point of the arc AB, prove that $CA.CB = CX.CD$.

12. AB is a diameter of a circle, centre O; AP, PQ are equal chords. Prove that $AP.PB = AQ.OP$.

13. TA, TB are the tangents from T to a circle; TPQ is a straight line cutting the circle at P and Q. Prove that (i) $AP : AQ = TP : TA$; (ii) $AP : AQ = BP : BQ$.

[14] E is a point inside the quadrilateral ABCD such that $\angle EAD = \angle BAC$ and $\angle EDA = \angle BCA$. Prove that

$$EB.AC = AB.DC.$$

15. ABC is a triangle. D is a point on the line bisecting $\angle BAC$ such that $AD^2 = AB.AC$. Prove that BD touches the circle ACD.

[16] A line PQ is divided at R so that $PR^2 = PQ.RQ$; TQR is a triangle such that $TQ = TR = PR$. Prove that

$$PT = PQ.$$

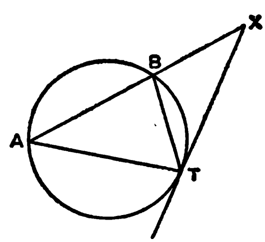


FIG. 874

17. If in fig. 874, the line through B parallel to AT cuts the tangent XT at K, prove that $XB^2 = XK.XT$.

18. In fig. 874, prove that

$$\frac{XA}{XB} = \frac{TA^2}{TB^2}.$$

[19] AB is a diameter of a circle, centre O. The tangents at A, B meet any other tangent at H, K. Prove that $AH.BK = AO^2$.

[20] In $\triangle ABC$, $\angle A = 90°$; $E$ is a point on $BC$ such that $AE = AB$. Prove that $BE.BC = 2AE^2$.

[Draw the perpendicular $AN$ from $A$ to $BC$.]

*21. $ABCD$ is a parallelogram; any line through $C$ cuts $DB$, $AB$, $DA$ produced, at $Q$, $Y$, $X$ respectively. Prove that

$$QX.QY = QC^2.$$

*22. The tangent at a point $C$ on a circle $PDE$ is parallel to a chord $DE$ and cuts two other chords $PD$, $PE$, when produced, at $A$, $B$. Prove that $AC : CB = AD : BE$.

*23. Two chords $AB$, $AC$ of a circle are produced to $P$, $Q$ so that $AB = BP$ and $AC = CQ$. If $PQ$ cuts the circle at $R$, prove that $AR^2 = PR.RQ$. [Let $BC$ cut $AR$ at $X$.]

*24. The tangent at $A$ to a circle $ABC$ cuts the line through $B$, parallel to $AC$, at $P$; the line through $C$ parallel to $AB$ cuts $AP$ at $Q$. Prove that $AP : AQ = AB^2 : AC^2$.

*25. An exterior common tangent to two circles touches them at $A$ and $C$; a variable line parallel to $AC$ cuts one circle at $P$ and the other at $Q$. Prove that the ratio $AP : CQ$ is constant. [Let $PQ$ cut the diameters $AB$, $CD$ of the circles at $H$, $K$. What do you know about $AP^2$?]

*26. $N$ is a point on the diameter $AB$ of a circle $AEB$ such that $AN = 4NB$; $E$ is the mid-point of the arc $AB$. If the perpendicular at $N$ to $AB$ cuts the arc $AEB$ at $K$ and cuts $AE$ produced at $Z$, prove that $ZK = KN$.

*27. $AB$ is a chord of a circle $APB$; the tangents at $A$, $B$ meet at $T$; $PH$, $PK$, $PZ$ are the perpendiculars to $TA$, $TB$, $AB$. Prove that $PH.PK = PZ^2$.

*28. $AC$, $CB$ are two sides of a regular decagon inscribed in a circle, centre $O$; $M$ is the mid-point of $AC$; $OM$ cuts $AB$ at $K$. Prove that

(i) $\angle BOK = 54°$; $\angle ACK = 18°$;
(ii) $BK.BA = BO^2$; $AK.AB = AC^2$;
(iii) $AC^2 + OB^2 = AB^2$.

*Therefore if* $p$ *and* $d$ *are the lengths of the sides of a regular pentagon and regular decagon inscribed in a circle, radius* $r$,

$$p^2 = d^2 + r^2.$$

## CONSTRUCTION 21

**Construct a mean proportional to two given lines.**

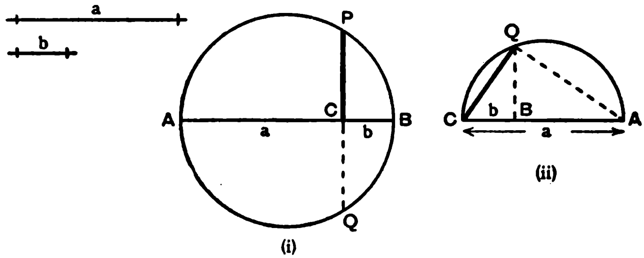



FIG. 875

**Given** two lines of lengths $a$, $b$ units.

**To Construct** a line of length $x$ units such that $x^2 = ab$.

### METHOD 1

**Construction.** Take a point C on a straight line and cut off from the line on **opposite** sides of C, parts CA, CB of lengths $a$, $b$ units; fig. 875 (i).
On AB as diameter, describe a circle.
Draw CP perpendicular to AB to cut the circle at P.
Then CP is the required mean proportional.

**Proof.** Produce PC to meet the circle at Q.

PQ is a chord perpendicular to the diameter AB.

$$\therefore \ PC = CQ.$$

But $\quad PC \cdot CQ = AC \cdot CB \quad$ *intersecting chords*,

$$\therefore \ CP^2 = AC \cdot CB$$
$$= ab.$$

### METHOD 2

**Construction.** Take a point C on a straight line and cut off from the line on the **same** side of C, parts CA, CB of lengths $a$, $b$ units; fig. 875 (ii).
On CA as diameter, describe a circle.
Draw BQ perpendicular to CA to cut the circle at Q.
Then CQ is the required mean proportional.

**Proof.** Join QA.

Since $\angle CQA = 90°$ $\angle$ *in semicircle,*
and since QB is the perpendicular from Q to CA, from the proof of Pythagoras' theorem,

$$CQ^2 = CA.CB$$
$$= ab.$$

*Or* as follows:

$$\angle CQA = 90° \qquad \angle \textit{ in semicircle,}$$

$\therefore$ CQ is a tangent to the circle on QA as diameter.

But $\angle QBA = 90°$,

$\therefore$ the circle on QA as diameter passes through B,

$$\therefore CQ^2 = CA.CB \qquad \textit{tangent property}$$
$$= ab.$$

NOTE. In practical constructions, Method 2 is often preferable to Method 1.

## EXERCISE 95

1. Construct a mean proportional between two lines of lengths 5 in. and 8 in. Measure it.

2. Construct a line of length $\sqrt{38}$ cm. and measure it. [Select values of $a$, $b$, for which $ab = 38$, which are fairly close together, *e.g.* 5 and $38 \div 5$, not 2 and 19.]

[3] Find graphically an approximate value of $\sqrt{31}$.

4. Draw a rectangle of sides 4 cm., 7 cm. and construct a square of equal area. Measure its side.

[5] Draw a rectangle of sides 2·6 in., 1·8 in. and construct a square of equal area. Measure its side.

6. Construct a square equal in area to a rhombus ABCD in which AB = 5 cm., $\angle A = 65°$. Measure its side.

[7] Construct a square equal in area to an equilateral triangle of side 6 cm. Measure its side.

[8] Solve graphically the equation, $(x - 3)^2 = 19$.

9. Construct a square equal in area to a quadrilateral ABCD in which AB = BC = 4 cm., CD = CA = 6 cm., AD = 7 cm. Measure its side.

[10] Construct a square equal in area to a regular pentagon, side 4 cm. Measure its side.

11. Draw a straight line ABC such that AB = 3 cm., BC = 5 cm. Draw AK perpendicular to AB and construct a point P on AK such that $AP^2 = AB.AC$. Construct a circle to pass through B, C and to touch AK. Measure its radius.

[12] Draw an angle AOB equal to 52°. Take two points H, K on OA such that OH = 1·5 in., OK = 2·5 in. and construct a circle to pass through H, K and touch OB. Measure its radius.

## Ratio of Areas of Similar Figures

**Examples for Oral Discussion**

1. (i) Draw a triangle ABC in which AB = 9 cm., BC = 7·5 cm., CA = 6 cm. How many triangles whose sides are 3 cm., 2·5 cm., 2 cm. can be cut out of it?

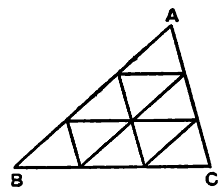


Fig. 876

(ii) XYZ is a triangle such that XY = 15 cm., YZ = 12·5 cm., ZX = 10 cm. How many triangles whose sides are 3 cm., 2·5 cm., 2 cm. can be cut out of it?

(iii) What is the ratio of (i) corresponding sides, (ii) areas of $\triangle$ABC, $\triangle$XYZ?

Nos. 2–4 refer to fig. 877 in which AH, XK are altitudes of the *similar* triangles ABC, XYZ.

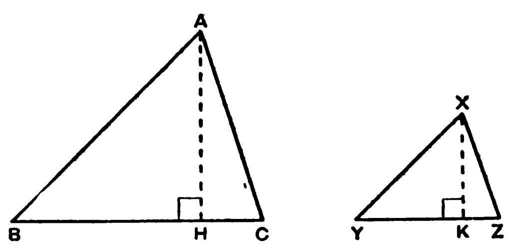


Fig. 877

2. If BC = 10 cm., YZ = 7 cm., AH = 8 cm., find XK and find the ratio $\triangle$ABC : $\triangle$XYZ.

**3.** If $BC : YZ = 5 : 4$, find the ratio $\triangle ABC : \triangle XYZ$.

(i) What is the value of $AB : XY$?

(ii) Explain why $\triangle$s $AHB$, $XKY$ are similar.

(iii) Complete the relation, $\dfrac{\triangle ABC}{\triangle XYZ} = \dfrac{\frac{1}{2}AH.BC}{\frac{1}{2}XK.YZ} = \ldots$

**4.** If $\triangle$s $\begin{matrix} ABC \\ XYZ \end{matrix}$ are similar, prove that

$$\triangle ABC : \triangle XYZ = BC^2 : YZ^2$$

and state the result in words.

Use the method indicated for No. 3.

**5.** If the polygons $\begin{matrix} ABCDE \\ PQRST \end{matrix}$ are similar, prove that

$$\text{area } ABCDE : \text{area } PQRST = AB^2 : PQ^2.$$

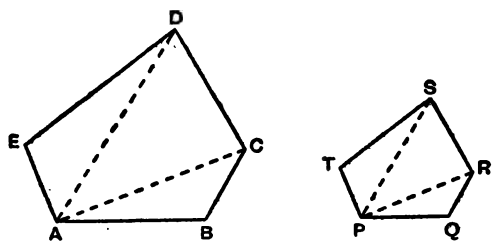



FIG. 878

It is given that

$$\angle A = \angle P,\ \angle B = \angle Q,\ \angle C = \angle R,\ \angle D = \angle S,\ \angle E = \angle T,$$

and that

$$AB : BC : CD : DE : EA = PQ : QR : RS : ST : TP.$$

Join $AC$, $AD$, $PR$, $PS$.

(i) Prove in succession that $\triangle$s $\begin{matrix} ABC \\ PQR \end{matrix}$, $\triangle$s $\begin{matrix} ACD \\ PRS \end{matrix}$, $\triangle$s $\begin{matrix} ADE \\ PST \end{matrix}$ are similar.

(ii) Prove that the ratio of the areas of these pairs of triangles is in each case equal to $AB^2 : PQ^2$.

The result established in No. 5 may be expressed as follows:

**The ratio of the areas of two similar polygons is equal to the ratio of the squares on corresponding sides.**

Hence it follows that

**The ratio of the areas of the surfaces of similar solids is equal to the ratio of the squares of corresponding lengths.**

And it can be proved that

**The ratio of the volumes of similar solids is equal to the ratio of the cubes of corresponding lengths.**

## NUMERICAL EXAMPLES

### EXERCISE 96

1. Find the ratio of the areas of the triangles in fig. 837, p. 485.

Find the ratio of the area of the quadrilateral to that of the smaller triangle in the following figures, Nos. 2–4:

2. Fig. 834, p. 485. [3] Fig. 835, p. 485. 4. Fig. 839, p. 485.

5. Find three integers to which the areas of the three triangles in fig. 830, p. 483, are proportional.

[6] In what ratio does HK divide △ABC in fig. 833, p. 485, (i) if BH = 3 in., (ii) if HK = 3 in.?

7. With the data of fig. 844 (ii), p. 487, write down the values of (i) △KAC : △KBD, (ii) △KAC : quad. ABDC.

[8] The lengths of the sides of △ABC are 10, 15, 20 cm. and the lengths of the sides of △PQR are 6, 9, 12 cm. Find the ratio of (i) their perimeters, (ii) their areas.

9. The areas of two similar triangles are 18 sq. in., 32 sq. in. The largest side of the first triangle is 9 in., find the largest side of the second triangle.

[10] A triangle ABC is divided by a line HK parallel to BC into two parts AHK, HKCB of areas 9 sq. cm., 16 sq. cm. respectively. If BC = 7 cm., find HK.

11. A screen 6 ft. high, not necessarily rectangular, requires 27 sq. ft. of material for covering. How much is needed for a screen of the same shape, 4 ft. high?

[12] On a map, scale 6 in. to the mile, a plot of a ground is represented by a quadrilateral of area $2\frac{1}{4}$ sq. in. Find the area of the plot in acres.

13. A light is 12 feet above the ground. Find the area of the shadow on the ground of the top of an oval table 4 feet high, and 45 sq. ft. in area.

[14] The area of the top of an oval table, 3 ft. high, is 20 sq. ft. and the area of its shadow on the floor cast by a lamp is 45 sq. ft. Find the height of the lamp above the floor.

15. If it costs £3 to gild a sphere of radius 3 ft., what will it cost to gild a sphere of radius 4 ft.?

[16] How many times can a cylindrical tumbler 4 in. high and 3 in. in diameter be filled from a cylindrical cask 40 in. high and 30 in. in diameter?

17. Two hot-water cans are the same shape; the smaller is 9 in. high and holds a quart; the larger is 15 in. high, how much will it hold?

18. A solid metal sphere, radius 3 in., weighs 8 lb. Find the weight of a solid sphere of the same metal 1 ft. in radius.

[19] A cylindrical tin 5 in. high holds $\frac{1}{4}$ lb. of tobacco. How much will a tin of the same shape 8 in. high hold?

20. Two models of the same statue are made of the same material, both being solid. One is 3 in. high and weighs 8 oz.; the other weighs 4 lb., find its height.

[21] A lodger pays 8 pence for a scuttle of coal, the scuttle being 20 in. deep. What would he pay for a scuttle of coal of the same shape, $2\frac{1}{2}$ feet deep?

*22. A tap can fill half of a spherical vessel, radius $1\frac{1}{2}$ feet, in 2 minutes. How long will two such taps take to fill one-quarter of a spherical vessel of radius 4 feet?

*23. Two leaden cylinders of equal lengths and diameters 3 in., 4 in., are melted and recast as a single cylinder of the same length. Find its diameter.

*24. ABC is a triangle in which $AB = AC = 2BC$; D is a point on AC such that $\angle DBC = \angle BAC$; a line through D parallel to BC cuts AB in E. Find the values of the ratios,

$$\triangle ABC : \triangle BCD : \triangle BED : \triangle EDA.$$

*25. Find the ratios of the areas of the 5 small triangles into which fig. 841, p. 486, is divided.

## THEOREM 83

**The ratio of the areas of two similar triangles is equal to the ratio of the areas of the squares on corresponding sides.**

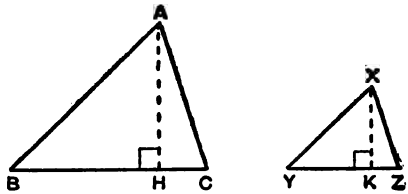


FIG. 879

**Given** two similar triangles ABC, XYZ.

**To prove that** $\dfrac{\triangle ABC}{\triangle XYZ}=\dfrac{BC^2}{YZ^2}.$

**Construction.** Draw the altitudes AH, XK.

**Proof.** Since AH, XK are altitudes,

$$\triangle ABC=\tfrac{1}{2}AH.BC,$$

and $$\triangle XYZ=\tfrac{1}{2}XK.YZ,$$

$$\therefore \frac{\triangle ABC}{\triangle XYZ}=\frac{\frac{1}{2}AH.BC}{\frac{1}{2}XK.YZ}=\frac{AH}{XK}\cdot\frac{BC}{YZ}.$$

But in the $\triangle$s ABH, XYK,

$\angle B=\angle Y$ *given,*

$\angle AHB=\angle XKY$ *rt. $\angle$s, constr.,*

$\therefore \angle BAH=\angle YXK$ *3rd $\angle$s of $\triangle$s.*

$\therefore \triangle$s $\begin{matrix}ABH\\XYK\end{matrix}$ are similar *equiangular $\triangle$s,*

$\therefore \dfrac{AH}{XK}=\dfrac{AB}{XY}$ *corr. sides proportional.*

But $\dfrac{AB}{XY}=\dfrac{BC}{YZ}$ *given,*

$$\therefore \frac{AH}{XK}=\frac{BC}{YZ},$$

$$\therefore \frac{\triangle ABC}{\triangle XYZ}=\frac{BC}{YZ}\cdot\frac{BC}{YZ}=\frac{BC^2}{YZ^2}.$$

The fact proved in Theorem 83 may also be deduced from the trigonometrical formula for the area of a triangle, see p. 246.

$$\triangle ABC = \tfrac{1}{2} AB . AC \sin A,$$
$$\triangle XYZ = \tfrac{1}{2} XY . XZ \sin X.$$

But $\angle A = \angle X$, $\therefore \dfrac{\triangle ABC}{\triangle XYZ} = \dfrac{AB}{XY} \cdot \dfrac{AC}{XZ}$.

But $\dfrac{AB}{XY} = \dfrac{AC}{XZ}$ *given*,

$$\therefore \frac{\triangle ABC}{\triangle XYZ} = \frac{AC^2}{XZ^2}.$$

This argument also shows that

**If two triangles ABC, XYZ are such that $\angle A = \angle X$,**

$$\frac{\triangle ABC}{\triangle XYZ} = \frac{AB}{XY} \cdot \frac{AC}{XZ}.$$

This result may also be proved by the method of Theorem 83, by drawing the altitudes BN, YP.

## EXERCISE 97

1. Two chords AB, CD of a circle, cut at X. Complete the relation, $\triangle AXD : \triangle BXC = AX^2 : \ldots$

[2] From a point X outside a circle, two lines XAB, XCD are drawn cutting the circle at A, B, C, D. Complete the relations, $\dfrac{\triangle BXD}{\triangle AXC} = \dfrac{XB^2}{\ldots} = \dfrac{XB . XD}{\ldots}$.

3. A line parallel to the side AB of $\triangle ABC$ meets AC produced, BC produced at P, Q. Complete the relations,

$$\frac{\triangle CAB}{\triangle CPQ} = \frac{CA^2}{\ldots} = \frac{AB . BC}{\ldots}.$$

4. ABCD is a square; APB, AQC are equilateral triangles. Find the ratio $\triangle APB : \triangle AQC$.

[5] ABC is a triangle such that

$$BC : CA : AB = 3 : 4 : 5.$$

If BPC, CQA, ARB are equilateral triangles, prove that

$$\triangle BPC + \triangle CQA = \triangle ARB.$$

6. The sides of a $\triangle ABC$ are trisected as shown in fig. 880. Prove that the area of $PQRSXY = \tfrac{2}{3} \triangle ABC$.

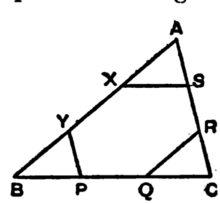



FIG. 880

[7] ABCD is a parallelogram; P, Q are the mid-points of CB, CD. Prove $\triangle APQ = \frac{3}{8}$ ||gram ABCD.

8. If in the $\triangle$s ABC, XYZ, $\angle A = \angle X$, prove that
$$\triangle ABC : \triangle XYZ = AB.AC : XY.XZ.$$

9. P, Q, R are points on the sides BC, CA, AB of $\triangle ABC$ such that $BP : PC = CQ : QA = AR : RB = 1 : 2$. Prove that $\triangle PQR = \frac{1}{3} \triangle ABC$.

[10] In $\triangle ABC$, $\angle A = 90°$ and AD is an altitude. Prove that $AB^2 : AC^2 = BD : DC$.

11. In $\triangle ABC$, $\angle A = 90°$ and AD is an altitude; DE is the perpendicular from D to AB. Prove that $BE : BA = BA^2 : BC^2$.

[12] Two circles cut at O. Straight lines AOP, BOS, COT cut one circle at A, B, C and cut the other at P, S, T. Prove that $\triangle ABC : \triangle PST = AB^2 : PS^2$.

13. ABC is a triangle inscribed in a circle. The tangent at C meets AB produced at T. Prove that

(i) $\triangle CAT : \triangle CBT = CA^2 : CB^2$; (ii) $CA^2 : CB^2 = AT : BT$.

14. In fig. 881, DPC is parallel to AB and $\angle ADC = \angle BCD = \angle APB$. Prove that
$$DP : PC = PA^2 : PB^2.$$

15. F is the mid-point of the side AB of $\triangle ABC$; P is a point on AB such that $AP^2 = AF.AB$. Prove that the line through P parallel to BC bisects the area of $\triangle ABC$.

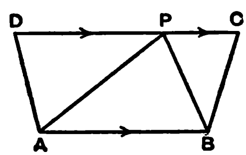


FIG. 881

*16. In $\triangle ABC$, $\angle A = 90°$; BCX, CAY, ABZ are similar triangles in which X, Y, Z are corresponding points. Prove that $\triangle CAY + \triangle ABZ = \triangle BCX$.

*17. AB is a diameter of a circle APB; AH, BK are the perpendiculars from A, B to the tangent at P. Prove that $\triangle APB = \frac{1}{2}$ quad. AHKB.

*18. AB is a diameter of a circle APB; BK is the perpendicular from B to the tangent at P; C is a point on AB such that $AC = AP$. If the line through C parallel to BP cuts AP at D, prove that $\triangle BKP =$ quad. BCDP.

*19. D, E, F are points on the sides BC, CA, AB of $\triangle ABC$ such that DEAF is a parallelogram. If $BD : DC = x : y$, prove that the ratio of the areas of DEAF to the area of ABC is equal to $2xy : (x+y)^2$.

# MISCELLANEOUS CONSTRUCTIONS

## CONSTRUCTION 22

**Construct a polygon similar to a given polygon ABCDE such that corresponding sides are in the given ratio XY : XZ.**

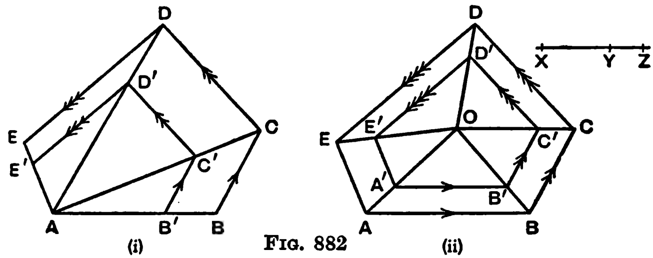



FIG. 882

In fig. 882 (i), AB is divided at B' so that

$$AB' : AB = XY : XZ$$

and arrows indicate that lines are drawn parallel.

In fig. 882 (ii), O is any point; OA is divided at A' so that

$$OA' : OA = XY : XZ$$

and arrows indicate that lines are drawn parallel.

In fig. 882 (i), AB'C'D'E' is the required polygon.

In fig. 882 (ii), A'B'C'D'E' is the required polygon.

The reader should perform the construction by each method and prove that it is correct.

### Similar Sections of a Pyramid

Two planes are called *parallel*, if they never meet one another. The reader should prove that if two parallel planes intersect a third plane, the lines of intersection are parallel.

If fig. 882 (ii) represents a pyramid, vertex O, base ABCDE, and if a plane parallel to the base cuts the edges at A', B', C', D', E', then the section A'B'C'D'E' is *similar* to the base ABCDE. Also

$$\text{area } A'B'C'D'E' : \text{area } ABCDE = A'B'^2 : AB^2 = OA'^2 : OA^2.$$

The proof is left to the reader.

## CONSTRUCTION 23

**Construct a triangle similar to a given triangle ABC and equal to a given fraction $\frac{XY}{XZ}$ of $\triangle ABC$.**

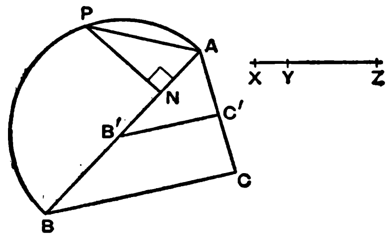



FIG. 883

(i) Divide AB at N so that

$$AN : AB = XY : XZ.$$

(ii) Draw NP perpendicular to AB and let it meet the circle on AB as diameter at P. Join AP.

(iii) From AB cut off AB′ equal to the mean proportional AP between AN, AB (see p. 514).

(iv) Draw B′C′ parallel to BC.

Then AB′C′ is the required triangle.

**Proof.**

$$\frac{\triangle AB'C'}{\triangle ABC} = \frac{AB'^2}{AB^2} = \frac{AN.AB}{AB^2} = \frac{AN}{AB} = \frac{XY}{XZ}.$$

NOTE. **A similar method may be used for any polygon.**

## CONSTRUCTION 24

**Construct a quadrilateral similar to a given quadrilateral ABCD and equal in area to a given rectangle XYZW.**

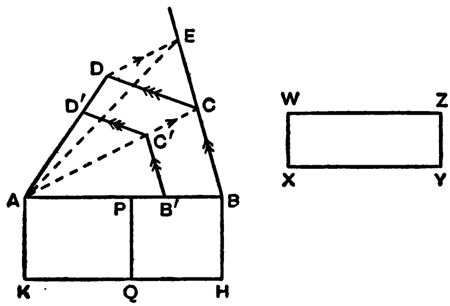



FIG. 884

(i) Construct rectangle ABHK equal in area to quad. ABCD.

(ii) Construct rectangle APQK equal in area to rect. XYZW.

(iii) Construct the mean proportional AB′ between AP and AB.

(iv) Construct quad. AB′C′D′ similar to quad. ABCD.

Then AB′C′D′ is the required quadrilateral.

For (i), start by reducing ABCD to the equivalent triangle ABE (p. 263).

For (ii), construct the fourth proportional AP to AK, XY, XW (p. 479).

**Proof.**
$$\frac{\text{quad. } AB'C'D'}{\text{quad. } ABCD} = \frac{AB'^2}{AB^2} = \frac{AP.AB}{AB^2}$$
$$= \frac{AP}{AB} = \frac{\text{rect. } APQK}{\text{rect. } ABHK}$$
$$= \frac{\text{rect. } XYZW}{\text{quad. } ABCD}.$$

$\therefore$ quad. AB′C′D′ = rect. XYZW.

## CONSTRUCTION 25

**Divide a given line AB at P so that $AB \cdot PB = AP^2$.**

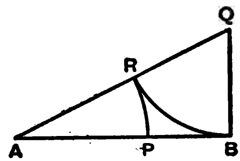


FIG. 885

(i) Draw BQ perpendicular to BA and equal to $\frac{1}{2}$AB.
(ii) Join QA and from it cut off QR equal to QB.
(iii) From AB cut off AP equal to AR.

Then P is the required point.

*The proof provides a useful exercise for the reader.*

Let $AB = 2l$ units, and $AP = AR = x$ units; explain why

$$(x+l)^2 = (2l)^2 + l^2.$$

Hence $$AP^2 = x^2 = 2l(2l - x) = AB \cdot PB.$$

The line AB is said to be divided at P in **medial section.**

The construction for dividing a line AB at P so that

$$AB \cdot PB = AP^2$$

is equivalent to the solution of the equation in $x$,

$$2l(2l - x) = x^2.$$

Therefore Construction 25 gives a geometrical solution of the quadratic equation,

$$x^2 + 2lx = 4l^2.$$

The reader should solve this equation by completing the square. This gives

$$x = l(\sqrt{5} - 1).$$

## CONSTRUCTION 26

**Construct $\triangle ABC$ given the length of $AB$ and that $\angle B = \angle C = 2\angle A$.**

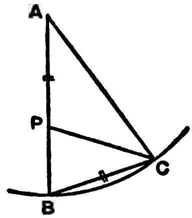



FIG. 886

(i) Draw a circle, centre A, radius AB.
(ii) Divide AB at P so that $AB . PB = AP^2$.
(iii) Place a chord BC in the circle so that $BC = AP$.

Then ABC is the required triangle.

*The proof provides a useful exercise for the reader.*

(*a*) Prove CB touches the circle CPA. [Theorem 82, p. 509.]
(*b*) Prove that $\triangle$s BCP, BAC are equiangular.
(*c*) Prove $CP = CB$ and $CP = AP$, and complete the proof.

### Regular Pentagon and Decagon

In Construction 26, the angles of $\triangle ABC$ are 36°, 72°, 72°. Therefore BC is the side of a regular decagon inscribed in the circle, centre A, radius AB.

If we use Construction 26 to inscribe a regular decagon in the circle, we obtain a regular pentagon by joining alternate vertices. It is, however, quicker to use another method, see p. 528.

If $AB = 2l$ units, $AP = l(\sqrt{5} - 1)$ units, see p. 526. But $BC = AP$; therefore the length of the sides of a regular decagon inscribed in a circle, radius $2l$ units, is $l(\sqrt{5} - 1)$ units.

By drawing a perpendicular from A to BC, we see that

$$\sin 18° = \cos 72° = \tfrac{1}{4}(\sqrt{5} - 1).$$

## Construction of Regular Pentagon

The quickest formal method for inscribing a regular pentagon in a given circle depends on the following property:—

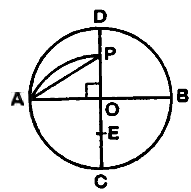
If $p$ and $d$ are the lengths of the sides of a regular pentagon and regular decagon inscribed in a circle, radius $r$, then $p^2 = d^2 + r^2$.

A method for proving this fact is indicated on p. 513, see No. 28.

Draw two perpendicular diameters AOB, COD of the given circle.

Bisect OC at E.

With centre E, radius EA, draw a circle to cut ED at P.




Then AP is equal to the sides of a regular pentagon inscribed in the circle.

(i) If the radius of the circle is $2l$ units, prove that $EA = l\sqrt{5}$ units.

(ii) Prove that OP is equal to the side of a regular decagon inscribed in the circle (see p. 527), and complete the proof.

## EXERCISE 98

**1.** Given a triangle ABC, construct a point P on BC such that the ratio of the perpendiculars from P to AB, AC is equal to 2 : 3.

[2] ABC is an equilateral triangle, side 5 cm.; construct a point P inside $\triangle$ABC such that the perpendiculars from P to BC, CA, AB are proportional to 1, 2, 3. Measure AP.

**3.** Construct an equilateral triangle ABC such that the length of the line joining A to a point of trisection of BC is 2 in. Measure BC.

[4] Construct a square ABCD such that the length of the line joining A to the mid-point of BC is 3 in. Measure AB.

5. Using a protractor, construct a regular pentagon ABCDE such that the distance of A from CD is 7 cm. Measure CD.

6. Given a quadrilateral ABCD, construct a similar quadrilateral PQRS such that PQ : AB = 3 : 5.

7. Given a triangle ABC, construct a square PQRS such that P, Q lie on AB, AC and RS lies along BC. [Start by drawing the square BCHK; join AH, AK.]

*The square* PQRS *is said to be inscribed in* $\triangle$ ABC.

[8] Inscribe in a given triangle a rectangle such that its height is half its base.

9. Inscribe in a given triangle a triangle whose sides are parallel to the sides of another given triangle.

[10] Given two radii OA, OB of a circle, construct a square PQRS such that P lies on OA, Q lies on OB and R, S lie on the arc AB.

11. ABC is an equilateral triangle, side 5 cm. Construct a line outside $\triangle$ ABC such that the lengths of the perpendiculars to it from A, B, C are proportional to 2, 3, 4. Measure the perpendicular from C.

12. Given a triangle ABC, construct a line parallel to BC cutting AB, AC at P, Q such that $\triangle APQ = \frac{1}{3}\triangle ABC$.

[13] Given a square ABCD, construct two lines parallel to AC which divide ABCD into three parts of equal area.

14. Given two equilateral triangles, construct an equilateral triangle whose area is the sum of their areas.

[15] Given two squares ABCD, PQRS and a line XY, construct a line ZW such that area ABCD : area PQRS = XY : ZW.

16. Construct an equilateral triangle ABC equal in area to a square, side 5 cm. Measure BC.

[17] Construct a triangle ABC such that BC : CA : AB = 6 : 5 : 4 and of area equal to a rectangle 4 cm. long, 3 cm. high. Measure BC.

18. Draw a quadrilateral ABCD in which $\angle A = 90°$, AB = 4 cm., BC = 6 cm., CD = 5·5 cm., DA = 3 cm. Construct a quadrilateral PQRS similar to ABCD and equal in area to a rectangle 4 cm. long, 3·5 cm. high. Measure QR.

[19] Using a protractor, construct a regular pentagon ABCDE equal in area to a square, side 6 cm. Measure AB.

*20. Construct an angle of 18°.

*21. Construct a circle to pass through two given points A, B and to touch a given circle S. [Draw any circle through A, B and cutting S at H, K say; let AB, HK meet at T; draw the tangents TP, TQ from T to S.]

*22. Construct a circle to pass through a given point A and touch a given circle S and have its centre on a given line BC. [Take the image A′ of A in BC.]

*23. Given four points A, B, C, D in order on a straight line, construct a point P on BC such that PA.PB = PC.PD.

*24. Construct a circle to touch two given lines AB, AC and touch a given circle, centre O, radius $a$. [Draw two lines parallel to AB, AC at distance $a$ from them; construct a circle to touch these lines and pass through O.]

*25. OA, OB are two lines such that OA = 6 cm., ∠AOB = 40°. Construct a circle touching OA at A and intercepting on OB a length of 5 cm.

*26. Draw a circle of radius 5 cm. and take a point A 3 cm. from the centre. Construct a chord PQ of the circle passing through A such that PA = $\frac{2}{3}$AQ.

## REVISION PAPERS 81–88 (Theorems 1–77)

[*Including Similar Triangles*]

*Arrows indicate that lines are given parallel.*

### 81

1. (i) How many angles each greater than 170° is it possible for a ten-sided convex polygon to have?
   (ii) What is the least number of obtuse angles a ten-sided convex polygon must have?

2. The tangent at C to a circle ABC and a chord DE of an intersecting circle ABDE meet when produced at T. If CAE is a straight line, prove that the circle through T, C, D passes through B.

3. (i) A line AB, 8 cm. long, is divided internally and externally in the ratio 3 : 1 at P and Q. Find the ratio PQ : AB.

(ii) Find the unknown marked lengths in fig. 887, the unit of length being 1 cm.

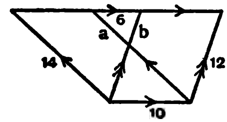



FIG. 887

4. If in fig. 888,

$$\angle ADC = \angle BEA = \angle CFB,$$

and if AD, BE, CF intersect at Y, Z, X as shown, prove that

$$YZ : BC = ZX : CA.$$

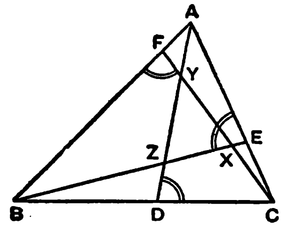



FIG. 888

## 82

1. Fig. 889 is a plan of a tennis court, measurements in feet. The only corner-markers that are visible are those at P and Q. Give the least necessary calculations you must make to find the exact position of A with the aid only of two tape measures.

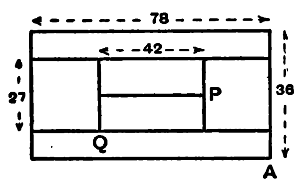



FIG. 889

2. AB, BC, CD are three equal chords of a circle. If the tangent at D meets BC produced at T, prove that

$$\angle BAD = \tfrac{2}{3}\angle TDA.$$

3. (i) Two intersecting lines AODF, BOCE are cut by three parallel lines AB, CD, EF. If AD = 7 in., DF = 3 in., CE = 4 in., EF = 2 in., AB = 3 in., find BC and CD.

(ii) Find the unknown marked lengths in fig. 890, the unit of length being 1 cm.

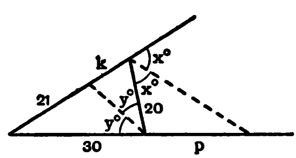



FIG. 890

4. APB, AQB are two circles. If PAQ is a straight line prove that BP : BQ is equal to the ratio of the diameters.

## 83

1. ABCD is a square in a horizontal plane; DK is a vertical post; P, Q are the mid-points of BA, BC; PQ cuts BD at R. If AB = 8 ft., DK = 7 ft., find the length of KR.

2. ABC is a given triangle. P is a variable point inside $\triangle$ ABC such that $\angle$PBA = $\angle$PCB. Find the precise locus of P.

3. Draw an equilateral triangle ABC, side 5 cm. Without making any more measurements, construct (i) a rectangle BCPQ equal to $\triangle$ABC, (ii) a rectangle QAXY equal to the rectangle BCPQ. State shortly your method.

4. AB is a diameter of the circle APB; Q is a point on the chord AP such that the perpendicular QN from Q to AB is equal to QP. Prove that AN : AP = QN : NB. [Join BQ, BP.]

## 84

1. K is any point on the diameter AB of a circle APB; P is the mid-point of the arc AB. Prove that $AK^2 + BK^2 = 2PK^2$.

2. ABCD is a parallelogram; P is any point on AC. If the circles PAD, PBC cut again at Q, prove that BQD is a straight line.

3. (i) Find the unknown marked lengths in fig. 891, the unit of length being 1 cm.

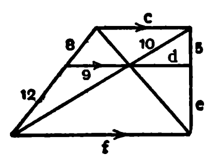



FIG. 891

(ii) A light is placed 4 ft. in front of a circular hole, diameter 3 in., in a partition. Find the diameter of the illuminated part of a wall 5 ft. behind the partition and parallel to it.

4. (i) AB, DC are parallel sides of the trapezium ABCD; AC cuts DB at K; the line through K parallel to AB cuts AD, BC at P, Q. Prove that PK = KQ.

(ii) ABCD, PQRS are quadrilaterals in which $\angle$A = $\angle$P, $\angle$B = $\angle$Q, $\angle$C = $\angle$R, and AB : BC = PQ : QR; prove that ABCD is similar to PQRS.

## 85

1. The sides BC, CA, AB of $\triangle$ ABC are produced their own lengths to X, Y, Z. Prove that $\triangle XYZ = 7\triangle ABC$.

2. The tangent at a point R of a circle meets a chord PQ, when produced, at T; O is the centre of the circle, and E is the mid-point of PQ. Prove that $\angle ROT = \angle RET$.

3. (i) ABCDE is a straight line such that AB = 3 in., BC = 2 in., CD = $2\frac{1}{2}$ in., DE = $1\frac{1}{2}$ in. Find the ratio in which (*a*) B divides AD, (*b*) E divides BD externally, (*c*) B divides CE externally.

   (ii) Prove that the triangle whose vertices are (2, 1), (5, 1), (4, 2) is similar to the triangle whose vertices are (1, 1), (7, 1), (5, 3).

4. P, Q are points in the side BC of $\triangle$ ABC such that BP = CQ. PH is drawn parallel to CA to cut BA at H; QK is drawn parallel to BA to cut CA at K. Prove that HK = BQ.

## 86

1. In $\triangle$ ABC, AB = 5 cm., AC = 3 cm., $\angle BAC = 120°$. Find the length of BC and the area of $\triangle$ ABC.

2. ABCD is a cyclic quadrilateral; AC cuts BD at K. If CD touches the circle KAD, prove that CB touches the circle KAB, and find two equal lines in the figure.

3. (i) In fig. 892, AF = FB and AD = 2DC. Find the values of the ratios,
   AE : EB; BH : HD;
   $\triangle$CHD : $\triangle$ABC.

   (ii) PQR is a triangle such that QR = 6 cm., RP = 4 cm., PQ = 5 cm. If K is a point on QR such that $\angle KPR = \angle PQR$, find the ratio QK : KR and the length of PK.

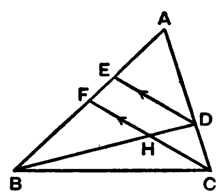


FIG. 892

4. (i) ABCD, PQRS are quadrilaterals in which $\angle A = \angle P$ and AB : BC : CD : DA = PQ : QR : RS : SP. Prove that ABCD is similar to PQRS.

   (ii) ABCD is a parallelogram; a line through A cuts BD, BC, DC roduced, at E, F, G respectively. Prove that AE : EF = AG : AF.

## 87*

1. In $\triangle$ ABC, AB = AC and $\angle$ BAC = 120°. If the perpendicular bisector of AB cuts BC at X, prove that BC = 3BX.

2. In fig. 893, PAQ, QKB, PMKN are straight lines. Prove that QM = QN. [Join AB, AN.]

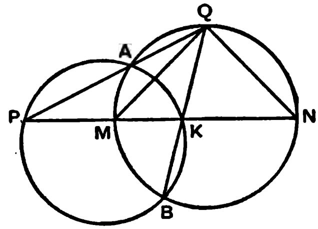


Fig. 893

3. (i) The diagonals AC, BD of the quadrilateral ABCD cut at K. The line through K parallel to AB cuts AD, BC at P, Q respectively. If AP = 10 cm., PD = 6 cm., PK = 9 cm., KQ = 4 cm., BQ = 9 cm., find AB and BC.

   (ii) A straight rod AB, 3 ft. 9 in. long, is fixed under water with A 2 ft. 6 in. and B 9 in. below the surface. Find the depth of a point C on the rod where AC = 1 ft.

4. ABCD is a quadrilateral; a line AF parallel to BC meets BD at F; a line BE parallel to AD meets AC at E. Prove that EF is parallel to CD.

## 88*

1. Two spheres, radii 6 in., 4 in., have their centres at distance 5 in. apart. Find the radius of the circle which is their curve of intersection and the distances of their centres from the plane of this circle.

2. The circles APQ, AHK touch one another at A. If PAH is a straight line and if the chords QP, HK meet, when produced, at a point T on the tangent at A, prove that T, A, Q, K are concyclic.

3. (i) A line HK parallel to AC meets AB, BC at H, K respectively and the bisector of $\angle$ ABC meets HK at N. If AH = 6 cm., HK = 15 cm., KC = 4 cm., find HN.

   (ii) P, Q, R are points on the sides BC, CA, AB respectively such that

   BP : PC = 4 : 5, CQ : QA = 3 : 1, AR : RB = 3 : 7.

   Find the ratio $\triangle$ PQR : $\triangle$ ABC.

4. ABCD is a straight line; O is a point outside it; a line through B parallel to OD cuts OA, OC produced at P, Q. If PB = BQ, prove that AB : BC = AD : CD.

## REVISION PAPERS 89–96 (Theorems 1–83)

[*Including intersecting chords of circle and ratio of areas*]

### 89

1. AB is a diameter of a circle APB; the tangent at A meets BP produced at Q. Prove that the tangent at P bisects AQ.

2. A circular cone is made from a sector of a circle of radius 6 in. and angle 240° Find the height of the cone.

3. XAY is a diameter of a circle, centre A; Z is the middle point of AY. If a circle is drawn on XZ as diameter, prove that the length of the tangent to this circle from any point P on the outer circle is equal to $\frac{1}{2}$PX. [Let PX cut the inner circle at Q; join QZ, PY.]

4. In fig. 894, ABC is an equilateral triangle and $\angle$PAQ = 120°; PBCQ is a straight line. Prove that

(i) PB.CQ = $BC^2$;
(ii) PB : CQ = $AP^2$ : $AQ^2$.

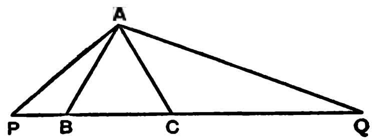


FIG. 894

### 90

1. In $\triangle$ABC, BC = 24 in., CA = 13 in., AB = 17 in. If BC is trisected at P, Q, find the lengths of AP, AQ.

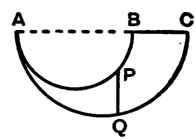


FIG. 895

2. In fig. 895, the circles touch at A; QP produced cuts AB at right angles at the centre of the larger circle. If PQ = 3 in., BC = 5 in., find the radius of each circle.

3. In fig. 896, prove that

(i) PQ : BC = NR : NC,
(ii) AN.BP = AB.QR.

If AX = 10 cm., XP = 5 cm., PB = 10 cm., AN = AC = 20 cm., prove that AR bisects $\angle$PAQ.

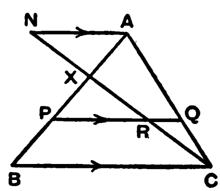


FIG. 896

4. A chord AD is parallel to a diameter BC of a circle; the tangent at C meets AD produced at E. Prove that BC.AE = $BD^2$.

## 91

1. PQRST is a variable pentagon. If the mid-points of the sides PQ, QR, RS, ST are fixed, prove that the side PT is of constant length and is fixed in direction.

2. AB, DC are the parallel sides of a trapezium ABCD; AC cuts BD at K. If the areas of $\triangle$AKB, $\triangle$AKD are 3 sq. in., 4 sq. in. respectively, find the area of $\triangle$DKC.

3. (i) With the data of fig. 897, find the values of $m$ and $p$.

(ii) Two straight lines XAB, XCD meet a circle at A, B, C, D. Prove that

XA . XD : XB . XC $=$ AD$^2$ : BC$^2$.

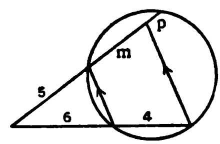


Fig. 897

4. P is any point on the circle ABCD; PH, PX, PK, PY are the perpendiculars from P to AB, BC, CD, DA, produced if necessary. Prove that

(i) $\triangle$s XPK, HPY are similar; (ii) PH . PK $=$ PX . PY.

[Join PC, PA.]

## 92

1. ABCD is a square; AB is produced to P so that AP $=$ AC; AN is the perpendicular from A to PD. Prove that PN $=$ 2ND.

2. The centre C of a circle ABP lies on a circle AQBC. If PAQ is a straight line and if QC produced cuts PB at R, prove that $\angle$PRQ $=$ 1 rt. $\angle$. [Join CA, CB, CP, QB.]

3. (i) ABCD is a rectangle; AB $=$ 8 in., BC $=$ 5 in.; P is a point inside ABCD whose distances from AD, AB are 2 in., 1 in. respectively; DP is produced to meet AB at E; CE cuts DA, when produced, at F. Find the lengths of EB and AF.

(ii) In $\triangle$ABC, AB $=$ 4 in., BC $=$ 3 in., CA $=$ 2 in. If D is a point on AB such that AD $=$ 1 in., prove that $\angle$ ACD $=$ $\angle$ ABC, and find the length of CD.

4. The bisector of $\angle$BAC of $\triangle$ABC cuts BC at D; CB is produced to K so that BK $=$ AC; AB is produced to cut the circle ADK at P. Prove that BP $=$ DC.

## 93

1. PH, PK are the perpendiculars from a point P to the lines AB, AC. If $\angle BAC = 30°$, prove that $AP = 2HK$.

2. In $\triangle ABC$, $\angle BAC = 1$ rt. $\angle$; the perpendicular bisector of BC cuts CA, BA produced at P, Q. Prove that BC touches the circle CPQ.

3. In fig. 898, TP is a tangent. Find the values of $q$, $r$, $t$.

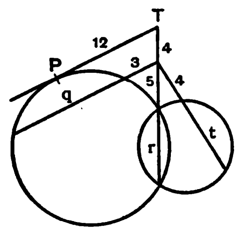



FIG. 898

4. (i) AB, DC are the parallel sides of the trapezium ABCD; AC cuts BD at E; DA, CB are produced to meet at F; EF cuts AB, DC at P, Q respectively. Prove that $QE : EP = QF : PF$.
   (ii) In $\triangle ABC$, I is the in-centre and $I_1$ is the ex-centre corresponding to BC. Prove that $AI . AI_1 = AB . AC$.

## 94

1. ABC is a triangle inscribed in a circle; P is any point on the minor arc BC; L, M, N are the feet of the perpendiculars from P to BC, CA, AB. Join PB, PC and prove that (i) P, B, L, N and P, C, L, M are concyclic, (ii) $\angle PLN = \angle PCA$, (iii) L, M, N are collinear.
*The straight line on which* L, M, N *lie is called the* **pedal line** (*or* **Simson line**) *of* P *with respect to* $\triangle ABC$.

2. AB is a diameter of the circle APB, centre O; the chord BP, when produced, cuts at T the tangent at A; OT meets the circle at Q. If $AT = 4$ cm., $QT = 2$ cm., find OQ and PT.

3. (i) The diagonals of the quadrilateral ABCD cut at K. State with reasons what angle in the figure equals $\angle DAC$ if (*a*) $AK . KC = BK . KD$, (*b*) $AK . KB = CK . KD$.
   (ii) AB is a given line, length 8 cm.; O is the mid-point of AB; P is a variable point on a circle, centre O, radius 6 cm. If PO produced meets the circle PAB at Q, find the locus of Q.

4. In $\triangle ABC$, $\angle A = 90°$; AD is an altitude of $\triangle ABC$. If the bisector of $\angle ABC$ meets AD at X and if the bisector of $\angle DAC$ meets BC at Y, prove that XY is parallel to AC.

## 95*

1. Two chords PQ, RS of a circle intersect at H; K is a point such that $\angle$KPQ and $\angle$KRS are right angles. Prove that HK is perpendicular to QS.

2. (i) A brick ABCD, 9 in. by 3 in., rests on the ground, and an equal brick PQRS is propped up against it as in fig. 899. If AP = 2 in., find the heights of Q, R, S above the ground.

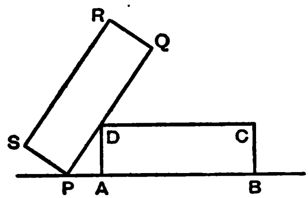


FIG. 899

(ii) A line HK parallel to BC cuts AB, AC at H, K; the distance between HK and BC is 5 cm. If the areas of AHK and HKCB are 9 sq. cm., 40 sq. cm., find the length of HK.

3. ABCD is a cyclic quadrilateral; BA, CD, when produced, meet at X; the line through X parallel to BC meets AD produced at E. Prove that $EX^2 = EA.ED$.

4. P, Q are points on the sides AC, AB of $\triangle$ABC such that $\angle BPC = \angle BQC$; BP cuts CQ at K; X, Y are points on AC, AB such that KXAY is a parallelogram. Prove that (i) $AX.XC = AY.YB$; (ii) the centre of the circle ABC is equidistant from X and Y.

## 96*

1. ABCD is a quadrilateral such that

$$\angle ABD = \angle DBC = \angle ADC = 45^\circ.$$

Prove that $\angle ADB = \frac{1}{2}\angle ACB$. [Draw CQ bisecting $\angle$ACB and cutting BD at Q; join AQ. Prove A, D, C, Q are concyclic.]

2. ABCD is a parallelogram; a line through A cuts BD, CD, BC produced, at P, Q, R respectively. Prove that $PQ : PR = PD^2 : PB^2$.

3. (i) In $\triangle$ABC, AB = AC; the bisector of $\angle$ABC meets AC at K. If the circle BAK cuts BC or BC produced again at D, prove that AK = CD.

(ii) Two circles, centres A and B, intersect at C, D; P is any point on CD. HPK is the chord of the circle, centre A, which is perpendicular to PA, and XPY is the chord of the circle, centre B, which is perpendicular to PB. Prove that HK = XY.

4. A is a fixed point on a given circle; AP, AQ are variable chords such that AP.AQ is constant. Prove that PQ touches a fixed circle, centre A.

# APPENDIX

## (I) PROOFS OF FUNDAMENTAL THEOREMS

### THEOREM 1

**If a straight line stands on another straight line, the sum of the adjacent angles so formed is equal to two right angles.**

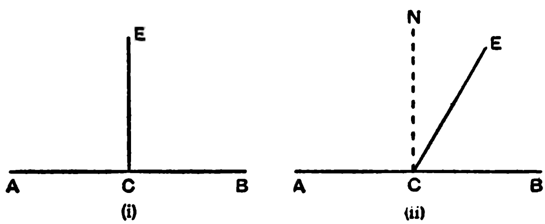



FIG. 900

**Given** a straight line CE meeting a straight line ACB.

**To prove that** $\angle ACE + \angle BCE = 2$ rt. $\angle$s.

*Case* 1. If $\angle ACE = \angle BCE$, each is by definition a right angle.

$\therefore \angle ACE + \angle BCE = 2$ rt. $\angle$s.

*Case* 2. If $\angle ACE$ is not equal to $\angle CEB$, suppose $\angle ACE$ is the greater.

**Construction.** Draw CN perpendicular to ACB.

**Proof.** $\angle ACE + \angle BCE = \angle ACN + \angle NCE + \angle ECB$

$= \angle ACN + \angle NCB$

$= 2$ rt. $\angle$s  *constr.*

If $\angle ECB$ is the greater, the proof is the same, except that A and B are interchanged.

## THEOREM 2

**If the sum of two adjacent angles is equal to two right angles, the exterior arms of the angles are in the same straight line.**

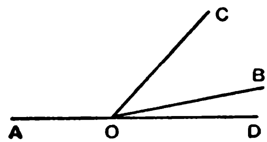


FIG. 901

**Given** two adjacent angles COA, COB such that

$$\angle COA + \angle COB = 2 \text{ rt. } \angle s.$$

**To prove that** AOB is a straight line.

**Construction.** Produce AO to D.

**Proof.** $\angle COA + \angle COD = 2$ rt. $\angle$s *adj. $\angle$s on st. line,*

but $\angle COA + \angle COB = 2$ rt. $\angle$s *given,*

$\therefore \angle COA + \angle COD = \angle COA + \angle COB$.

From these equals, take away the common $\angle COA$,

$\therefore \angle COD = \angle COB$.

But $\angle COB$, $\angle COD$ are on the same side of OC,

$\therefore$ OD is the same straight line as OB.

But AOD is a straight line *constr.,*

$\therefore$ AOB is a straight line.

## THEOREM 4

**If two triangles have two sides of the one equal to two sides of the other, each to each, and also the angles included by those sides equal, the triangles are congruent.**

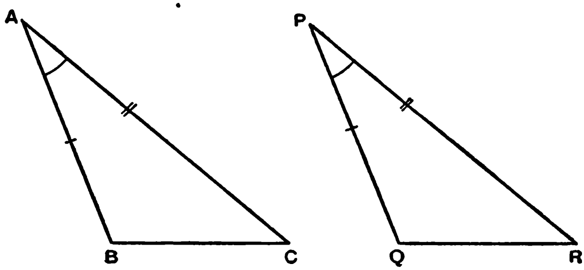


FIG. 902

**Given** two triangles ABC, PQR such that

$$AB = PQ, \quad AC = PR, \quad \angle BAC = \angle QPR.$$

**To prove that** $\triangle s \begin{smallmatrix} ABC \\ PQR \end{smallmatrix}$ are congruent.

**Proof.** Apply the triangle ABC to the triangle PQR so that A falls on P and the line AB falls along the line PQ and C falls on the same side of PQ as R.

Since $AB = PQ$ *given*,
B falls on Q.

Since AB falls along PQ and $\angle BAC = \angle QPR$ *given*,
AC falls along PR.

Since $AC = PR$ *given*,
C falls on R.

Since B falls on Q and C falls on R,
BC coincides with QR,

$\because$ $\triangle ABC$ coincides with $\triangle PQR$,

$\therefore$ $\triangle s \begin{smallmatrix} ABC \\ PQR \end{smallmatrix}$ are congruent.

## INEQUALITY THEOREM

**If one side of a triangle is produced, the exterior angle is greater than either of the interior opposite angles.**

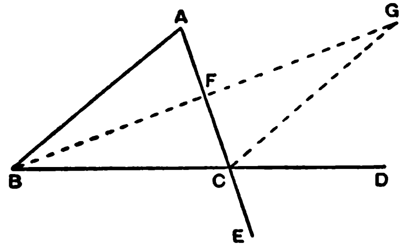



FIG. 903

**Given** a triangle ABC with BC produced to D.

**To prove that** $\angle ACD > \angle BAC$ and $\angle ACD > \angle ABC$.

**Construction.** Bisect AC at F.

Join BF and produce it to G so that $BF = FG$.

Join CG.

**Proof.** In the triangles CFG, AFB,

$CF = AF$ *constr.*,

$GF = BF$ *constr.*,

$\angle CFG = \angle AFB$ *vert. opp.* $\angle$*s*,

$\therefore$ $\triangle$s $\begin{matrix} CFG \\ AFB \end{matrix}$ are congruent SAS.

$\therefore \angle FCG = \angle BAF$.

But $\angle ACD$ is greater than its part $\angle FCG$,

$\therefore \angle ACD > \angle BAF$ or $\angle BAC$.

Similarly, if BC is bisected at H and if AH is produced to K so that $AH = HK$, it can be proved that

$\angle BCE > \angle ABC$.

But $\angle ACD = \angle BCE$ *vert. opp.* $\angle$*s*.

$\therefore \angle ACD > \angle ABC$.

NOTE. This theorem may be used to prove Theorem 5.

### THEOREM 5

**Two straight lines are parallel if a transversal makes a pair of alternate angles equal.**

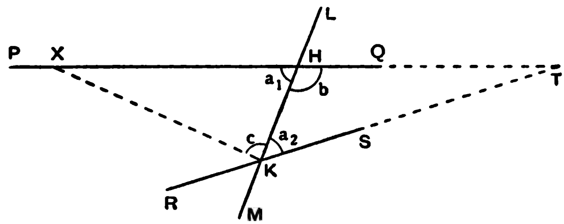



FIG. 904

**Given** a straight line LHKM cutting two straight lines PHQ, RKS such that $\angle$PHK = alternate $\angle$HKS.

**To prove that** PQ is parallel to RS.

**Construction and Proof.** If PQ, RS are not parallel, they will meet when produced, either towards Q and S or towards P and R. Suppose if possible that PQ and RS, when produced towards Q and S, meet at T.

Take a point X on TH produced so that HX = KT. Join KX.

In $\triangle$s XHK, TKH, XH = TK *constr.*,

KH = HK

$a_1 = a_2$ *given*,

$\therefore$ $\triangle$s $\begin{matrix}\text{XHK}\\ \text{TKH}\end{matrix}$ are congruent SAS.

$\therefore\ c = b.$

But $a_2 = a_1$ *given*, $\therefore\ c + a_2 = b + a_1.$

But $b + a_1 = 2$ rt. $\angle$s *adj. $\angle$s on st. line*,

$\therefore\ c + a_2 = 2$ rt. $\angle$s.

But these are adjacent angles, $\therefore$ XKT is a straight line.

$\therefore$ the straight lines XHT, XKT coincide, and this is contrary to what is given.

$\therefore$ PQ, RS cannot meet when produced towards Q, S.

Similarly, it can be proved that QP, SR cannot meet when produced towards P, R.

$\therefore$ PQ is parallel to RS.

## THEOREM 6

**If a transversal cuts two parallel straight lines,**
**(i) alternate angles are equal,**
**(ii) corresponding angles are equal,**
**(iii) interior angles on the same side of the transversal are supplementary.**

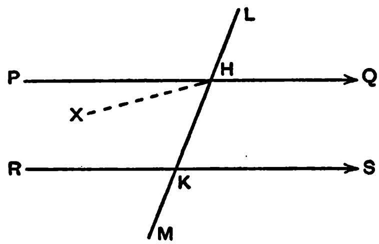



FIG. 905

**Given** two parallel lines PHQ, RKS and a transversal LHKM.

**To prove that** (i) $\angle PHK = \angle HKS$,
(ii) $\angle LHQ = \angle HKS$,
(iii) $\angle QHK + \angle HKS = 2$ rt. $\angle$s.

(i) **Construction.** If $\angle PHK$ is unequal to $\angle HKS$, draw HX so that $\angle XHK$ is equal to the alternate $\angle HKS$.

**Proof.** $\angle XHK =$ alternate $\angle HKS$ — *constr.*,
$\therefore$ XH is parallel to KS,
but PH is parallel to KS — *given*,
$\therefore$ two intersecting lines XH, PH are both parallel to KS, but this is impossible by Playfair's Axiom.
$\therefore$ $\angle PHK$ cannot be unequal to $\angle HKS$,
$\therefore$ $\angle PHK = \angle HKS$.

(ii) $\angle LHQ = \angle PHK$ — *vert. opp.* $\angle$*s*,
$= \angle HKS$ — *proved*,

(iii) $\angle QHK + \angle PHK = 2$ rt. $\angle$s — *adj.* $\angle$*s on st. line*,
but $\angle PHK = \angle HKS$ — *proved*,
$\therefore$ $\angle QHK + \angle HKS = 2$ rt. $\angle$s.

## THEOREM 10

**If two triangles have two angles of the one equal to two angles of the other, each to each, and also a side of one equal to the corresponding side of the other, the triangles are congruent.**

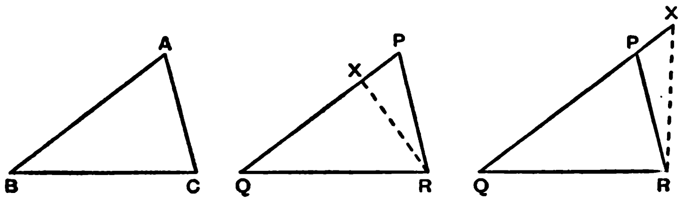



FIG. 906

**Given** two triangles ABC, PQR such that

$$BC = QR$$

and the angles of two of the pairs

$$\angle A,\ \angle P;\ \angle B,\ \angle Q;\ \angle C,\ \angle R$$

are equal.

**To prove that** $\triangle$s $\begin{matrix}ABC\\PQR\end{matrix}$ are congruent.

**Construction and Proof.** Since the sum of the angles of a triangle is two right angles, the angles of the third pair are also equal.

Suppose, if possible, that QP is not equal to BA, then there is a point X on QP or QP produced such that QX = BA.

Join RX.

In $\triangle$s ABC, XQR,

| | |
|---|---|
| $BC = QR$ | *given*, |
| $BA = QX$ | *constr.*, |
| $\angle B = \angle Q$ | *given*, |

$\therefore$ $\triangle$s $\begin{matrix}ABC\\XQR\end{matrix}$ are congruent SAS.

$\therefore \angle ACB = \angle XRQ$;

but $\angle ACB = \angle PRQ$ *given or proved,*

$\therefore \angle PRQ = \angle XRQ$;

but this is impossible because one of these angles is a part of the other.

$\therefore$ QP cannot be unequal to BA.

$\therefore QP = BA$.

$\therefore$ in $\triangle$s ABC, PQR,

$AB = PQ$ *proved,*

$BC = QR$ *given,*

$\angle B = \angle Q$ *given,*

$\therefore \triangle$s $\begin{matrix} ABC \\ PQR \end{matrix}$ are congruent SAS.

## THEOREM 54

**In equal circles, equal angles at the centres and equal angles at the circumferences stand on equal arcs.**

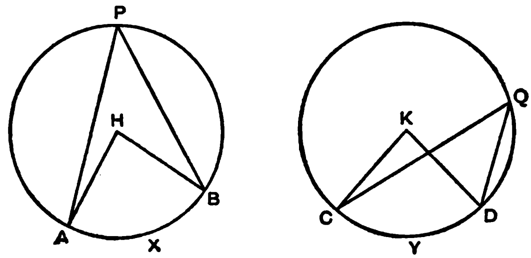



FIG. 907

(i) **Given** two equal circles AXBP, CYDQ, centres H, K, and two arcs AXB, CYD which subtend equal angles AHB, CKD at the centres.

**To prove that** arc AXB = arc CYD.

**Proof.** Apply the circle AXB to the circle CYD so that the centre H falls on the centre K and HA falls along KC and HB falls on the same side of KC as KD.

Since the circles are equal, A falls on C and the circumferences coincide.

Since $\angle AHB = \angle CKD$ *given*,

HB falls along KD and B falls on D.

$\therefore$ the arcs AXB, CYD coincide.

$\therefore$ arc AXB = arc CYD.

(ii) **Given** two equal circles ABP, CDQ, centres H, K, and two arcs AXB, CYD which subtend equal angles APB, CQD at the circumferences.

**To prove that** arc AXB = arc CYD.

**Proof.** $\angle AHB = 2\angle APB$   *$\angle$ at centre = twice $\angle$ at* $O^{ce}$.

$\angle CKD = 2\angle CQD$   *$\angle$ at centre = twice $\angle$ at* $O^{ce}$,

but $\angle APB = \angle CQD$   *given*, $\therefore \angle AHB = \angle CKD$.

$\therefore$ arc AXB = arc CYD.

## THEOREM 55

**In equal circles, equal arcs subtend equal angles at the centres and equal angles at the circumferences.**

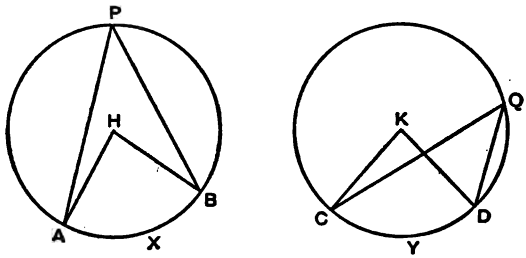



FIG. 908

**Given** two equal circles AXBP, CYDQ, centres H, K, and two equal arcs AXB, CYD.

**To prove that** (i) $\angle AHB = \angle CKD$,
(ii) $\angle APB = \angle CQD$.

(i) **Proof.** Apply the circle AXB to the circle CYD so that the centre H falls on the centre K and HA falls along KC and HB falls on the same side of KC as KD. Since the circles are equal, A falls on C and the circumferences coincide.

Since arc AXB = arc CYD *given*,

B falls on D and HB falls on KD,

$\therefore$ $\angle AHB$ coincides with $\angle CKD$,

$\therefore$ $\angle AHB = \angle CKD$.

(ii) **Proof.** $\angle AHB = 2\angle APD$ *$\angle$ at centre = twice $\angle$ at* O^ce^,
$\angle CKD = 2\angle CQD$ *$\angle$ at centre = twice $\angle$ at* O^ce^.

But $\angle AHB = \angle CKD$ *proved*,

$\therefore$ $\angle APB = \angle CQD$.

## (II) THE TANGENT AS A LIMITING CHORD

A is a given point on a circle. Any line AQ through A cuts the circle at P.

Bisect arc AP at $P_1$; bisect arc $AP_1$ at $P_2$; bisect arc $AP_2$ at $P_3$, etc. We can repeat this bisection process as often as we like, and thus obtain a succession of lines $AP_1Q_1$, $AP_2Q_2$, $AP_3Q_3$, etc., which cut off from the circle arcs of continually decreasing lengths.

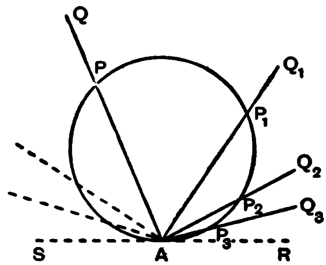



FIG. 909

However often we repeat the process we cannot obtain a line cutting off an arc of zero length; but by repeating it sufficiently often we can obtain a line which cuts off an arc as short as we please, and all further lines obtained will cut off still shorter arcs.

The limiting position AR of this series of lines is called the **tangent at A** and is a line drawn through A cutting off an arc of zero length.

If the process of repeated bisection is performed on the other side of the chord AP, we obtain the limiting position AS of this series of lines, which is equally by our definition the tangent at A. It is necessary to prove that AR and AS are in one straight line, in order to show that there is only one tangent to a circle at any point A. This is done by showing that each is at right angles to the radius through A.

## THEOREM 59

**The tangent to a circle is perpendicular to the radius drawn through the point of contact.**

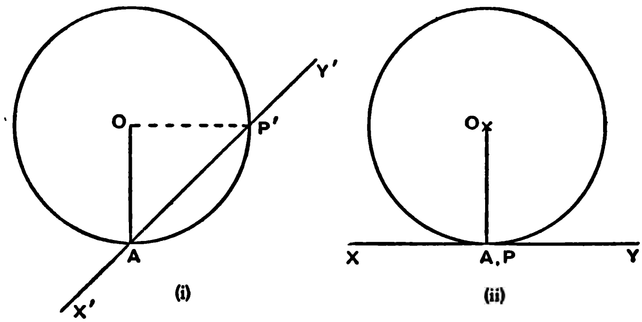



FIG. 910

**Given** a tangent XAY at a point A on a circle, centre O.

**To prove that** $\angle$OAY is a right angle.

**Construction.** Through A draw any line X'AP'Y' cutting the circle again at P'.
Join OP'.

**Proof.** OA = OP' *radii*,

$\therefore$ $\angle$OAP' = $\angle$OP'A *base $\angle$s, isos. $\triangle$.*

Since X'AP'Y' is a straight line, $\angle$OAX', $\angle$OP'Y' are the supplements of $\angle$OAP', $\angle$OP'A,

$\therefore$ $\angle$OAX' = $\angle$OP'Y'.

Now the tangent XAY at A is the limiting position of the line X'AP'Y' when the arc AP' is decreased without limit so that P' coincides with A.

$\therefore$ $\angle$OAX = $\angle$OAY.

But these are adjacent angles on a straight line,

$\therefore$ $\angle$OAY = 1 rt. $\angle$.

## THEOREM 61

**If a straight line touches a circle and from the point of contact a chord is drawn, the angles which the chord makes with the tangent are equal to the angles in the alternate segments of the circle.**

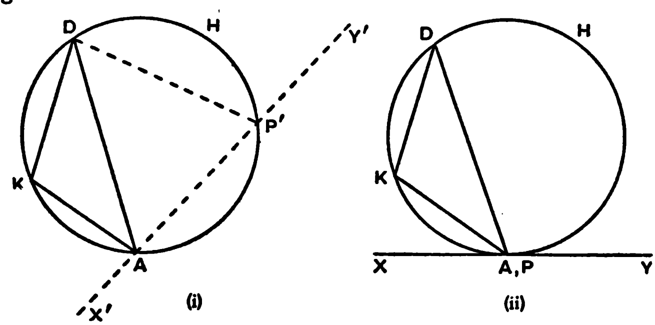



FIG. 911

**Given** a tangent XAY at a point A on a circle and a chord AD forming the two segments AHD, AKD.

**To prove that** (i) $\angle$DAY $=\angle$AKD in alternate segment AKD,

(ii) $\angle$DAX $=\angle$AHD in alternate segment AHD.

**Construction.** Through A draw any line X′AP′Y′ cutting the arc AHD at P′.
Join DP′.

**Proof.** $\angle$DP′Y′ $=\angle$AKD *ext.* $\angle$ *of cyclic quad.* $=$ *int. opp.* $\angle$.

Now the tangent at A is the limiting position of the line X′AP′Y′ when the arc AP′ is decreased without limit so that P′ coincides with A.

But the limiting position of $\angle$DP′Y′ is $\angle$DAY.

$\therefore$ when X′AP′Y′ becomes the tangent XAY at A,

$\angle$DAY $=\angle$AKD.

Similarly, it may be proved that $\angle$DAX $=\angle$AHD.

## (III) SUMMARY OF THEOREMS †

**Theorem 1** (pp. 86, 540)

If **ACB** is a straight line,

$$a+b=2 \text{ rt. } \angle\text{s}.$$

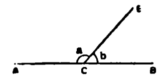


FIG. 204

**Theorem 2** (pp. 87, 541)

If $a+b=2$ rt. $\angle$s,

**AOB** is a straight line.

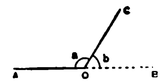


FIG. 206

**Theorem 3** (pp. 28, 87)

If two straight lines intersect, the vertically opposite angles are equal.

$$a=b \quad \text{and} \quad p=q.$$

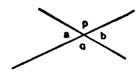


FIG. 207

**Theorem 4** (p. 542)

If **AB** = **PQ**, **AC** = **PR**, $\angle$**A** = $\angle$**P**,

$\triangle$s $\begin{matrix}\text{ABC}\\ \text{PQR}\end{matrix}$ are congruent   SAS.

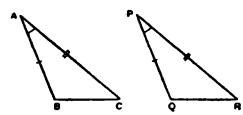


FIG. 281

**Theorem 5** (pp. 95, 544)

**PQ** and **RS** are parallel if

*either* $a=b$   *alt.* $\angle$*s*,
*or* $c=b$   *corr.* $\angle$*s*,
*or* $b+d=2$ rt. $\angle$s   *int.* $\angle$*s*.

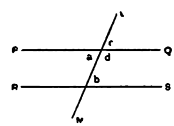


FIG. 224

**Theorem 6** (p. 545)

If **PQ** and **RS** are parallel,

$a=b$   *alt.* $\angle$*s*,
$c=b$   *corr.* $\angle$*s*,
$b+d=2$ rt. $\angle$s   *int.* $\angle$*s*.

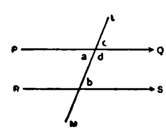


FIG. 225

† The figures in this Summary are small reproductions of the figures in the main text; reference is given in each case to the page on which the larger-scale figure will be found.

**Theorem 7** (pp. 39, 96)

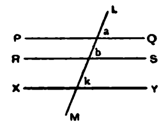



FIG. 79

If PQ and RS are each parallel to XY, then PQ is parallel to RS.

**Theorem 8** (p. 102)

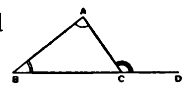



FIG. 100

(i) If the side BC of $\triangle$ABC is produced to D,

$$\angle ACD = \angle A + \angle B.$$

(ii) $\angle A + \angle B + \angle ACB = 2$ rt. $\angle$s.

**Theorem 9** (p. 112)

(i) The sum of the *interior* angles of a convex polygon with $n$ sides is $(2n-4)$ rt. $\angle$s.

(ii) The sum of the *exterior* angles of a *convex* polygon with $n$ sides formed by producing the sides in order is 4 rt. $\angle$s.

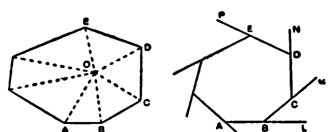



FIG. 275

**Theorem 10** (p. 546)

If $\angle B = \angle Q$ and $\angle C = \angle R$ and if *either* BC = QR *or* AB = PQ *or* AC = PR,

$\triangle$s $\begin{matrix} ABC \\ PQR \end{matrix}$ are congruent     ASA, AAS.

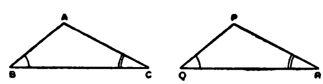



FIG. 282

**Theorem 11** (p. 119)

If AB = AC,
then $\angle B = \angle C$.

**Theorem 12** (p. 120)

If $\angle B = \angle C$,
then AB = AC.

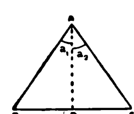
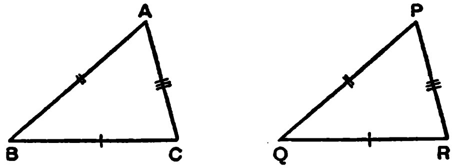



FIG. 289

**Theorem 13** (p. 128)




If $AB = PQ$, $BC = QR$, $CA = RP$,

$\triangle$s $\begin{matrix} ABC \\ PQR \end{matrix}$ are congruent SSS.

**Theorem 14** (p. 130)

If $AC = XZ$, $AB = XY$, and if $\angle B$, $\angle Y$ are right angles,

$\triangle$s $\begin{matrix} ABC \\ XYZ \end{matrix}$ are congruent RHS.

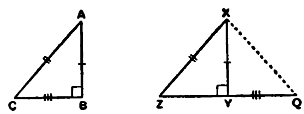



FIG. 318

**Theorem 15** (p. 150)

If $ABCD$ is a parallelogram,

(i) $AB = DC$ and $AD = BC$;
(ii) $\angle A = \angle C$ and $\angle ABC = \angle ADC$;
(iii) $BD$ bisects area $ABCD$.

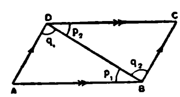



FIG. 349

**Theorem 16** (p. 151)

If $ABCD$ is a parallelogram whose diagonals cut at $K$,

$AK = KC$ and $BK = KD$.

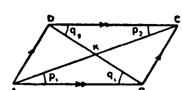



FIG. 350

**Theorem 17** (p. 152)

If $ABCD$ is a quadrilateral in which $AB$ is equal and parallel to $DC$, then $ABCD$ is a parallelogram.

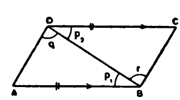



FIG. 351

**Theorem 18 (p. 153)**

If ABCD is a quadrilateral in which

$\angle A = \angle C$ and $\angle B = \angle D$,

then ABCD is a parallelogram.

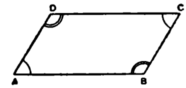


FIG. 352

**Theorem 19 (p. 154)**

If ABCD is a quadrilateral in which

$AB = DC$ and $AD = BC$,

then ABCD is a parallelogram.

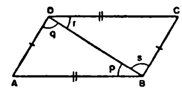


FIG. 353

**Theorem 20 (p. 155)**

If the diagonals of a quadrilateral ABCD cut at K and if $AK = KC$ and $BK = KD$, then ABCD is a parallelogram.

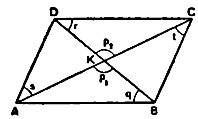


FIG. 354

**Theorem 21 (p. 167)**

If in $\triangle ABC$, $AC > AB$,

then $\angle ABC > \angle ACB$.

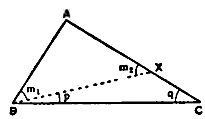


FIG. 371

**Theorem 22 (pp. 168, 169)**

If in $\triangle ABC$, $\angle B > \angle C$,

then $AC > AB$.

FIG. 372

**Theorem 23 (p. 170)**

If CN is the perpendicular from C to a straight line ANPB,

then $CN < CP$

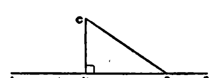


FIG. 376

**Theorem 24** (p. 172)

If ABC is any triangle,

$$BA + AC > BC.$$

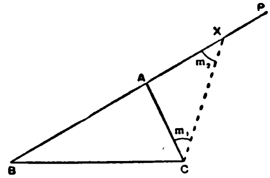



FIG. 377

**Theorem 25** (p. 180)

If H, K are the mid-points of AB, AC,

then (i) HK is parallel to BC,

(ii) $HK = \frac{1}{2}BC$.

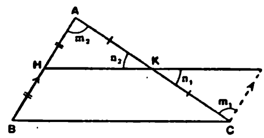



FIG. 397

**Theorem 26** (p. 181)

If H is the mid-point of AB, and if the line through H parallel to BC cuts AC at K,

then $AK = KC$.

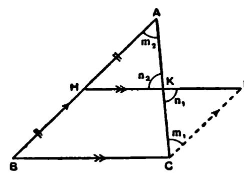



FIG. 398

**Theorem 27** (p. 182)

If two transversals ABCDE, PQRST are cut by the parallel lines

BQ, CR, DS, ET,

and if $BC = CD = DE$,

then $QR = RS = ST$.

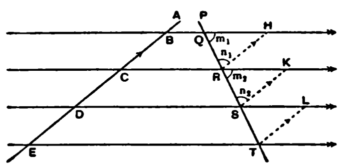



FIG. 399

**Theorem 28** (p. 192)

The medians AD, BE, CF of $\triangle ABC$ concur at a point G, such that

$$DG = \tfrac{1}{3}DA, \quad EG = \tfrac{1}{3}EB, \quad FG = \tfrac{1}{3}FC.$$

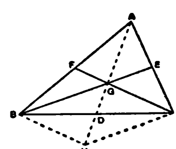



FIG. 416

**Theorem 29** (pp. 200, 201)

The locus of a point P equidistant from two given points A and B is the perpendicular bisector of AB.

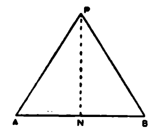



Fig. 419

**Theorem 30** (p. 204)

The perpendicular bisectors of the three sides of a triangle are concurrent.

The point at which they concur is the *circumcentre* of the triangle.

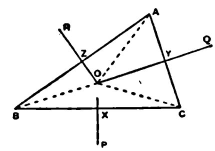



Fig. 424

**Theorem 31** (p. 205)

The altitudes of a triangle are concurrent.

The point at which the altitudes concur is called the *orthocentre* of the triangle.

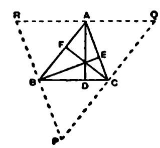



Fig. 425

**Theorem 32** (pp. 208, 209)

The locus of a point which is equidistant from two given intersecting straight lines is the pair of lines which bisect the angles between the given lines.

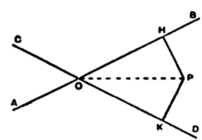



Fig. 428

**Theorem 33** (p. 213)

The internal bisectors of the three angles of a triangle are concurrent.

The point at which they concur is the *in-centre* of the triangle.

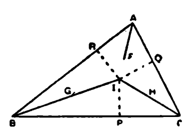



Fig. 434

**Theorem 34** (p. 235)

The area of a rectangle is measured by the product of the measures of two adjacent sides.

**Theorem 35** (p. 252)

The area of a parallelogram ABCD is equal to the area of a rectangle ABHK on the same base AB and between the same parallels AB, KHDC.

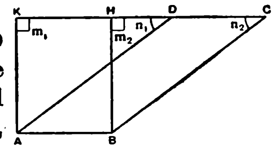



FIG. 498

**Corollary 1.** Parallelograms on the same base and between the same parallels are equal in area.

**Corollary 2.** Area of parallelogram = base × height.

**Corollary 3.** Parallelograms on equal bases and between the same parallels are equal in area.

**Theorem 36** (p. 254)

The area of a triangle ABC is equal to half the area of a rectangle PQBC on the same base BC and between the same parallels BC, QPA.

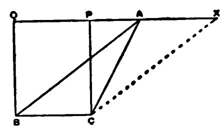



FIG. 499

**Corollary 1.** Area of triangle = $\frac{1}{2}$ base × height.

**Corollary 2.** Triangles on the same or equal bases and of equal altitudes are equal in area.

**Corollary 3.** If triangles of the same area have the same or equal bases, their altitudes are equal.

**Theorem 37** (p. 256)

If AD is parallel to BC,

$\triangle$ABC = $\triangle$DBC.

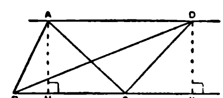



FIG. 500

**Theorem 38** (p. 257).

If $\triangle$ABC = $\triangle$DBC and if A, D are on the same side of BC, then AD is parallel to BC.

**Theorem 39 (p. 258)**

If a triangle ABC and a parallelogram PQBC are on the same base BC and between the same parallels BC, AQP,

$\triangle ABC = \frac{1}{2}$||gram PQBC.

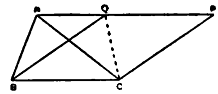


FIG. 502

**Theorem 40 (p. 282)**

If $\angle BAC = 1$ rt. $\angle$,

and if AX is the perpendicular from A to BC,

(i) $BA^2 = BX.BC$ and $CA^2 = CX.CB$,

(ii) $AB^2 + AC^2 = BC^2$.

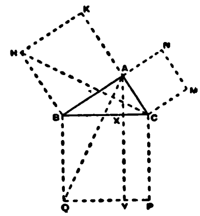


FIG. 546

**Theorem 41 (p. 285)**

If in $\triangle ABC$,

$AB^2 + BC^2 = AC^2$,

then $\angle ABC = 1$ rt. $\angle$.

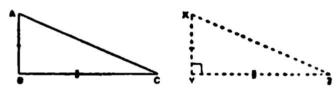


FIG. 548

**Theorem 42 (p. 300)**

If M is the mid-point of a chord AB of a circle, centre O,

then $\angle OMA = 1$ rt. $\angle$.

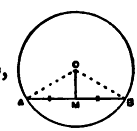


FIG. 564

**Theorem 43 (p. 301)**

If ON is the perpendicular from the centre O of a circle to a chord AB,

then $AN = NB$.

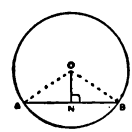


FIG. 565

**Theorem 44 (p. 302)**

If the chords AB and CD of a circle are equal, they are equidistant from the centre.

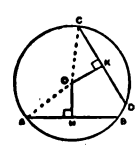


FIG. 566

**Theorem 45** (p. 303)

If the chords AB and CD of a circle are equidistant from the centre,

then $$AB = CD.$$

**Theorem 46** (p. 304)

There is one circle, and only one circle which passes through three given points A, B, C, not in the same straight line.

The perpendicular bisectors of BA, BC meet at the centre O of the circle.

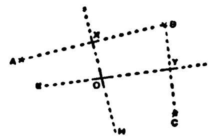


FIG. 568

**Theorem 47** (p. 313)

The angle which an arc of a circle subtends at the centre is double that which it subtends at any point on the remaining part of the circumference. If O is the centre of the circle APB,

$$\angle AOB = 2\angle APB.$$

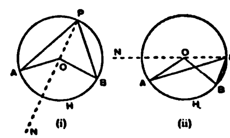


FIG. 583 (i), (ii)

**Theorem 48** (p. 320)

If APQB is a circle,

$$\angle APB = \angle AQB.$$

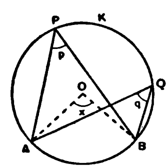


FIG. 596 (i) *only*

**Theorem 49** (p. 321)

If AB is a diameter of the circle APB,

$$\angle APB = 1 \text{ rt. } \angle.$$

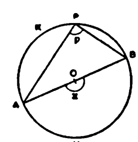


FIG. 597

**Theorem 50** (p. 322)

(i) If ABCD is a circle,

$\angle B + \angle D = 2$ rt. $\angle$s.

(ii) If ABCD is a circle and if the chord AD is produced to E,

$\angle CDE = \angle ABC$.

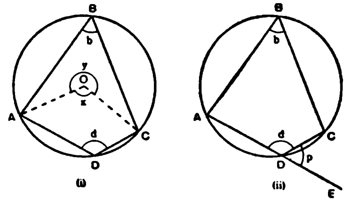


FIG. 598

**Theorem 51** (p. 331)

If $\angle BAC = 1$ rt. $\angle$,

the circle on BC as diameter passes through A.

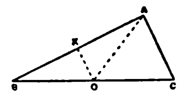


FIG. 620

**Theorem 52** (p. 332)

If $\angle APB = \angle AQB$, and if P, Q are on the same side of AB,

A, B, P, Q are concyclic.

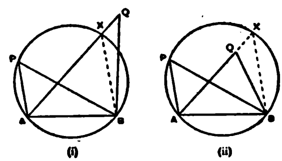


FIG. 616

**Theorem 53** (p. 334)

If ABCD is a quadrilateral in which $\angle ABC + \angle ADC = 2$ rt. $\angle$s,

A, B, C, D are concyclic.

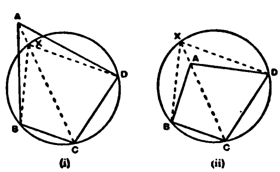


FIG. 619

**Theorem 54** (p. 548)

If H, K are the centres of two equal circles APBX, CQDY,

and if $\angle AHB = \angle CKD$,

*or* if $\angle APB = \angle CQD$,

then arc AXB = arc CYD.

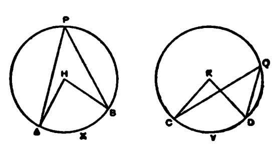


FIG. 907

**Theorem 55** (p. 549)

If H, K are the centres of two equal circles APBX, CQDY, and if arc AXB = arc CYD,

then $\angle$AHB = $\angle$CKD

and the angles at the circumferences standing on the equal arcs AXB, CYD are equal.

**Theorem 56** (p. 347)

If AB and CD are equal chords of equal circles,

then minor arc AB = minor arc CD.

**Theorem 57** (p. 348)

If AB and CD are equal arcs of equal circles,

then chord AB = chord CD.

**Theorem 58** (p. 358)

If OA is a radius of a circle, centre O, and if BAC is a straight line perpendicular to OA,

BAC is a tangent to the circle.

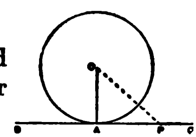


FIG. 659

**Theorem 59** (pp. 359, 551)

If BAC is the tangent at A to a circle, centre O,

$\angle$OAB = 1 rt. $\angle$.

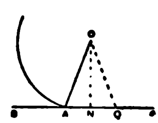


FIG. 660

**Theorem 60** (p. 360)

If TP, TQ are the tangents from T to a circle, centre O, P and Q being the points of contact,

(i) TP = TQ;

(ii) $\angle$TOP = $\angle$TOQ;

(iii) OT bisects $\angle$PTQ.

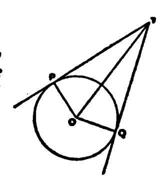


FIG. 661

**Theorem 61** (pp. 368, 552)

If BAC is the tangent at A to a circle and if AD is any chord,

$\angle DAC = \angle APD$

in alternate segment.

$\angle DAB = \angle AQD$

in alternate segment.

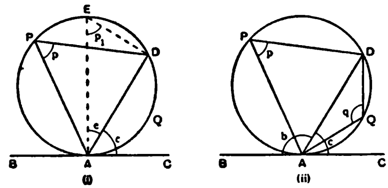



FIG. 680

**Theorem 62** (p. 370)

If P and C are points on opposite sides of AD such that $\angle DAC = \angle APD$,

AC touches at A the circle APD.

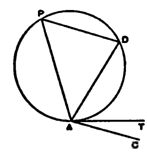



FIG. 681

**Theorem 63** (p. 379)

If two circles, centres A, B, touch one another at P, then A, P, B are collinear.

If the contact is external,

AB = sum of radii.

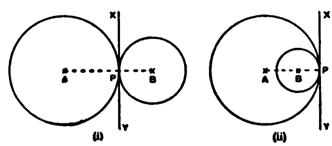



FIG. 699

If the contact is internal, AB = difference of radii.

**Theorem 64** (p. 417)

If CN is an altitude of $\triangle ABC$, and if $\angle BAC$ is obtuse,

$$BC^2 = BA^2 + CA^2 + 2BA \cdot AN.$$

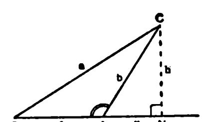



FIG. 749

**Theorem 65** (p. 418)

If CN is an altitude of $\triangle ABC$, and if $\angle BAC$ is acute,

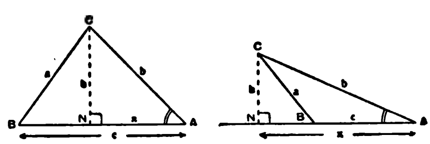



FIG. 750

$$BC^2 = BA^2 + CA^2 - 2BA \cdot AN.$$

**Theorem 66** (p. 419)

If AD is a median of $\triangle$ABC,

$$AB^2 + AC^2 = 2AD^2 + 2BD^2.$$

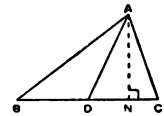


FIG. 751 (ii)

**Theorem 67** (p. 426)

If the chords AB, CD of a circle, centre O, radius $r$, intersect at a point X inside the circle,

$$XA\,.\,XB = XC\,.\,XD = r^2 - OX^2.$$

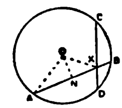


FIG. 761

**Theorem 68** (p. 427)

If the chords AB, CD of a circle, centre O, radius $r$, intersect at a point X outside the circle, and if XT is the tangent from X to the circle,

$$XA\,.\,XB = XC\,.\,XD = XT^2 = OX^2 - r^2.$$

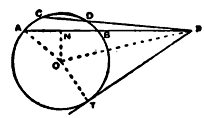


FIG. 762

**Theorem 69** (p. 428)

(i) If AB and CD intersect at X or meet, when both are produced, at X, and if XA . XB = XC . XD, then A, B, C, D are concyclic.

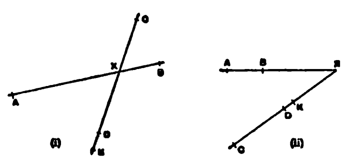


FIG. 763

(ii) If AB is produced to X and if C is a point not on AB such that

$$XA\,.\,XB = XC^2,$$

then the circle ABC touches XC at C.

**Theorem 70** (p. 462)

If the altitudes AH, XK of $\triangle$ABC, $\triangle$XYZ are equal,

$$\frac{\triangle ABC}{\triangle XYZ} = \frac{BC}{YZ}.$$

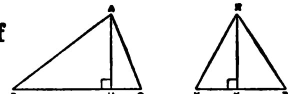


FIG. 806

**Theorem 71 (p. 464).**

If a line parallel to BC cuts AB, AC at X, Y,

then $\frac{AX}{XB} = \frac{AY}{YC}$.

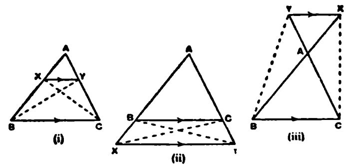



FIG. 808

**Theorem 72 (p. 465)**

If X, Y are points on AB, AC or on AB, AC produced such that $\frac{AX}{XB} = \frac{AY}{YC}$, then XY is parallel to BC.

**Theorem 73 (p. 472)**

If the internal or external bisector of $\angle$BAC meets BC or BC produced at D,

then $\frac{BD}{DC} = \frac{AB}{AC}$.

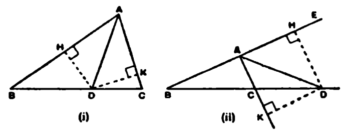



FIG. 821

**Theorem 74 (p. 473)**

If the base BC of $\triangle$ABC is divided internally or externally at D so that $\frac{BD}{DC} = \frac{AB}{AC}$,

then AD is an internal or external bisector of $\angle$BAC.

**Theorem 75 (p. 490)**

If in $\triangle$s ABC, XYZ,

$\angle A = \angle X$, $\angle B = \angle Y$, $\angle C = \angle Z$,

then $\frac{AB}{XY} = \frac{BC}{YZ} = \frac{CA}{ZX}$.

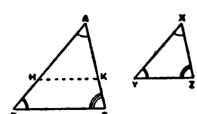



FIG. 853

**Theorem 76 (p. 492)**

If in $\triangle$s ABC, XYZ,

$$\frac{BC}{YZ} = \frac{CA}{ZX} = \frac{AB}{XY},$$

then

$\angle A = \angle X$, $\angle B = \angle Y$, $\angle C = \angle Z$.

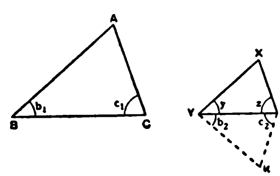



FIG. 828

**Theorem 77** (p. 496)

If in $\triangle$s ABC, XYZ,

$$\frac{AB}{XY} = \frac{AC}{XZ} \text{ and } \angle A = \angle X,$$

then $\angle B = \angle Y$ and $\angle C = \angle Z$.

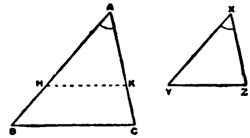


FIG. 829

**Theorem 78** (p. 504)

If AD is an altitude of $\triangle$ABC, and if $\angle$BAC = 1 rt. $\angle$, then

(i) $\triangle$s ABC, DBA, DAC are similar;

(ii) $AD^2 = BD . DC$;

(iii) $BA^2 = BD . BC$ and $CA^2 = CD . CB$.

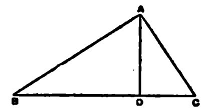


FIG. 866

Theorems 79–82 are equivalent to Theorems 67–69. For proofs, by using similar triangles, see pp. 506–509.

**Theorem 83** (p. 520)

If $\triangle$s $\begin{matrix} ABC \\ XYZ \end{matrix}$ are similar,

$$\frac{\triangle ABC}{\triangle XYZ} = \frac{BC^2}{YZ^2}.$$

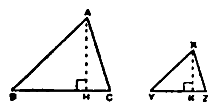


FIG. 877

## (iv) SUMMARY OF CONSTRUCTIONS

1. Fundamental elementary constructions. Pages 135–142.

2. Construction of a square; construction with numerical data of parallelogram, trapezium, etc. Pages 162, 163.

3. Division of a line into equal parts. Page 188.

4. Reduction of a quadrilateral and a polygon to an equivalent triangle. Page 263.

5. Construction of circle through three points. Page 304.

6. Construction of a tangent to a circle. Pages 384, 385.

7. Construction of common tangents. Pages 386, 387.

8. Construction of inscribed and escribed circles. Pages 388, 389.

9. Construction of a segment of a circle. Page 390.

10. Construction for inscribing a triangle in a given circle and circumscribing a triangle about a given circle, equiangular to a given triangle. Page 392.

11. Construction of circles through given points or touching given lines or circles. Pages 392–394, 434.

12. Construction of a square equivalent to a rectangle or polygon. Pages 432, 433.

13. Division of a line in a given ratio. Page 478.

14. Construction of third & fourth proportionals. Page 479.

15. Construction of mean proportional. Pages 514, 515.

16. Construction of a polygon similar to a given polygon and with (i) the sides, (ii) the areas, in a given ratio. Pages 523, 524.

17. Construction of a quadrilateral similar to a given quadrilateral and equivalent to a given rectangle. Page 525.

18. Division of a line in medial section. Page 526.

19. Construction of a regular pentagon and a regular decagon. Pages 527, 528.

# INDEX

*(See also Table of Contents and Summaries of Theorems and Constructions)*

# ANSWERS

## STAGE B

# ANSWERS

## STAGE B

## *PART I*

**Page 88** EXERCISE 24

1. 160°, 30°, 88°. 2. 36. 3. 22. 4. 144°. 5. 6, 11, 22.
6. 180°, 6°. 7. 270°, 10°. 8. 150°, 100°. 9. 60°.
10. 240°. 11. 45°. 12. $157\frac{1}{2}$°. 13. $112\frac{1}{2}$°. 14. 135°.
15. $157\frac{1}{2}$°. 16. 135°. 17. 140°. 18. 137°. 19. 120°.
20. 120°. 21. 210°. 22. 45°. 23. 20. 24. 15. 25. 24.
26. 95. 27. 50. 28. 120°. 29. 135°. 30. 135°. 31. 105°.
32. 40°, 50°. 33. $x+y+z=180$. 34. 126°, 63°.
35. PAS, QAT. 36. 110; KAM, HAL.

**Page 92** EXERCISE 25

1. $a+b$, $b+2c+d$. 2. $b+c+d$, $c$. 3. $\angle$ROT, $\angle$POS.
4. $c=d$, $a+b+c+d=180°$. 5. $b=d$, $b+c=90°$

**Page 97** EXERCISE 26

1. 24. 2. 55. 3. 20. 4. 36, 72. 5. 22, 57.
6. 30°. 7. 80° 8. 120°. 9. 85°. 11. AQ, BY; AP, BZ.

**Page 104** EXERCISE 28

1. 65, 80°. 2. 65, 50°. 3. 35, 75°. 4. 47°.
5. 35°; $90°-x°$. 6. 147°. 7. 76°. 8. 66°. 9. 45°.
10. 80°. 11. $90°-\frac{1}{2}$C. 12. $90°-$A. 13. No, yes, no.
14. 20. 15. 40. 16. 110°. 17. 110°. 18. 40°.
19. $x=36$, $z=72$. 20. 25°. 21. 38°, 86°. 22. 10°. 25. 130°.
26. 92°. 27. 53°, 70°, 57°. 28. 58°. 29. 84°.

**Page 106** EXERCISE 29

8. $b+c-s$. 12. $p-a+b$. 13. $c-a-b$. 14. $f-d-e$. 15. $\angle$B.

**Page 108** ORAL EXAMPLES

1. 8 rt. $\angle$s. 2. 10 rt. $\angle$s. 3. 16, 196, $2n-4$, rt. $\angle$s.
4. 4 rt. $\angle$s. 5. 4 rt. $\angle$s. 6. 4, 4, rt. $\angle$s. 7. 4 rt. $\angle$s.
8. $p+q+r-s+t=4$ rt. $\angle$s.

**Page 110** EXERCISE 30

1. 70°. 2. 122°. 3. 80°. 4. 18. 5. 54. 6. 80.
7. 36°. 8. 56, 76, rt. $\angle$s. 10. 45°, 36°.

11. 24°, $\frac{4}{n}$ rt. ∠s. 12. 9, 10. 13. 24, 18. 14. 17.
15. 60°. 16. 6 rt. ∠s. 17. 30°. 19. 72°, 111°, 108°, 69°.
20. 10 rt. ∠s. 21. 20. 24. 250°, 110°.

**Page 116** EXERCISE 32

1. $\frac{ABC}{FED}$; SAS. 2. No. 3. $\frac{ABC}{EDF}$; ASA. 4. No.
5. $\frac{ABC}{EFD}$; AAS. 6. No. 7. No. 8. ∠X, CA.
17. △BAP, △CAQ. 20. △QXC; △XDQ, △XPB; ∠XDQ.

**Page 122** EXERCISE 33

1. 35°. 2. 56°. 3. $x$, $180-2x$ or $90-\frac{1}{2}x$, $90-\frac{1}{2}x$, degrees.
4. 72°, 72°, 36°. 5. 36°, 36°, 108°. 6. 30°. 7. $57\frac{1}{2}$°.
8. 35°, 125°, 20°. 9. 36°. 11. 38°. 16. $360-2y$.
17. $(45-\frac{1}{4}z)$ degrees. 18. $3x-180$. 19. $4(90-x)$ degrees. 22. 36°.

**Page 149** EXERCISE 37

1. 58°. 2. 23°. 3. 62°. 4. 110°. 5. 55°, 35°. 6. 32°.
7. 125°. 8. 18°, 27°. 9. 30°, 30°. 11. $22\frac{1}{2}$°. 12. 54°.
13. 126°, 30°, 24°. 14. 35°, 115°, 30°. 15. 150°. 16. 72°.
17. $67\frac{1}{2}$°. 18. $22\frac{1}{2}$°, 135°, $22\frac{1}{2}$°.

**Page 160** EXERCISE 39

1. $36\frac{3}{4}$°; impossible. 2. 2·59. 3. 2·93. 4. 4·79; impossible.
5. 11·3. 6. 6·68; 5·66, 3·53; impossible. 7. 8·87. 8. 8·41.
9. $104\frac{1}{2}$°. 10. 4·96. 11. 6·76. 12. $63\frac{1}{4}$°. 13. 5·18. 14. 3·82.
15. $62\frac{1}{2}$°. 16. 5·23 cm. 17. $49\frac{1}{2}$°. 18. $82\frac{1}{2}$°, $8\frac{3}{4}$°.

**Page 162** ORAL EXAMPLES

1. 6·78 cm. 2. 7·36 cm. 3. 3·55 cm.

**Page 164** EXERCISE 40

1. 4·47 cm. 2. $97\frac{1}{4}$°. 3. 2·55 cm. 4. 3·54 cm.
5. 7·13, 3·63, cm. 6. $106\frac{1}{4}$°. 7. 5·74 cm. 8. 4·08 in.
9. 5·41 cm. 10. 3·39 in. 11. 6·09 cm. 12. $25\frac{1}{4}$°. 13. 8·25.
14. 5·34. 15. 6·21. 16. 6. 17. 8·64 cm. 19. $52\frac{1}{2}$°.
20. $117\frac{1}{4}$°. 21. 3·08 cm. 22. 8·24 cm. 25. 4·26 cm.
27. 4·26. 28. 9·28 or 3·72. 29. 10·08 or 2·92. 30. 7·67.
31. 4·78. 32. 7·82. 33. 8·71. 34. 6·22 cm.

**Page 173** EXERCISE 41

1. AB. 2. IB. 3. PC, QC. 4. RB, PB.
5. AB, PC. 6. CX, AX, BX. 7. PB.
8. BP, PA, AB. 9. No; yes; no, vertices collinear. 10. 7.
11. 7 cm., 5 cm. 12. BD.

**Page 174** EXERCISE 42

9. CB. 18. XC.

**Page 178** EXERCISE 43

1. 1·8, 1·5, in. 2. 3·5, 2·5, 3, in. 3. 1·8, 1·6, in.
4. AQ $=\frac{3}{7}$AC, PQ $=\frac{4}{7}$BC. 5. 9, 8. 6. $7\frac{1}{2}$. 7. 18.
8. 16, 7·5, cm. 9. 6 in. 10. 5·5 in. 11. 1·5 in. 12. 5 in.
13. $8\frac{1}{2}$ ft. 14. No; height 5 ft.

**Page 187** EXERCISE 44

26. 17 in., 12 in.

**Page 190** ORAL EXAMPLES

8. (i) 0·01, 0·04, 0·07, 0·09, in.; (ii) 2·31, 1·84, 2·47, 2·59, in.

**Page 193** EXERCISE 45

6. 5·74 cm. 7. 8·61 cm. 8. 10·3 cm.

**Page 206** EXERCISE 47

1. 3·57 cm. 7. 4·75 in.

**Page 216** EXERCISE 48

21. 5 in., $1\frac{1}{4}$ in., from XK.
25. (i) a circle, centre O; (ii) an ellipse. 31. 4; 1.

REVISION PAPERS, 1–34

**Page 218**

*Paper* 1. 1. 30°, 15°. 2. 2 in. 3. 130°, 93°. 4. 53°.
*Paper* 2. 1. 163°, $3\frac{1}{2}$ rt. ∠s. 2. $\frac{1}{2}$ in. 3. 99°.

**Page 219**

*Paper* 3. 1. 36°, 135°. 3. 4·67 cm.
*Paper* 4. 1. 46°. 2. 10·73 cm. 3. 36°. 4. $b+c+d$.
*Paper* 5. 1. Q, A, M; 75°; R, A, N. 2. 36°. 3. 8.

**Page 220**

*Paper* 6. **1.** $6\frac{2}{3}$ rt. $\angle$s. **2.** 108. **3.** $90 - x - y$, $x \sim y$, degrees.
*Paper* 7. **1.** 60°. **3.** 60.

**Page 221**

*Paper* 8. **1.** 36, 288°. **2.** 50°, 72°

**3.** $y = \dfrac{6x}{8 - x}$; $x = 4, 5, 6, 7$; $y = 6, 10, 18, 42$.

*Paper* 9. **1.** 68°.

**Page 222**

*Paper* 10. **1.** 72°. **2.** $a + b - c$. **3.** 36°.
*Paper* 11. **1.** 110°.
*Paper* 12. **2.** 120°; AE, CD; ED, BC.

**Page 223**

*Paper* 13. **1.** $180 - 4x$, $\frac{1}{2}x$, degrees. **2.** $a + b - c$.
*Paper* 14. **1.** 72°. **2.** 26 rt. $\angle$s.
*Paper* 15. **1.** 20.

**Page 224**

*Paper* 16. **2.** 108°.
*Paper* 17. **2.** 54°

**Page 225**

*Paper* 18. **1.** 65. **2.** 3·26 in.
*Paper* 19. **1.** 85°.
*Paper* 20. **2.** 54°.

**Page 226**

*Paper* 21. **1.** $67\frac{1}{2}$°. **2.** $45\frac{1}{4}$°.
*Paper* 22. **2.** 4·10 cm.

**Page 227**

*Paper* 23. **1.** 18°.
*Paper* 24. **1.** 1·57 in.

**Page 228**

*Paper* 25. **3.** RS; SP.
*Paper* 27. **2.** 7·62 cm. **3.** QP

**Page 230**

*Paper* 29. **2.** 30°.

**Page 231**

*Paper* 31. **3.** BA, AQ, QB.

**Page 232**

*Paper* 34. **2.** 3·73 cm.

## *PART II* (*Section* 1)

**Page 234** ORAL EXAMPLES

1. $4\frac{3}{8}$ sq. in.; 6·3 sq. in.; 3·6 sq. in.; $2\frac{1}{16}$ sq. in.
2. $\frac{1}{16}$ sq. mi.; 0·04 sq. mi.; 0·2 sq. mi.; 8 sq. in.

**Page 237** EXERCISE 49

1. 3·22 sq. in. 3. 20 sq. in. 4. 72 sq. in. 5. 36 sq. in.
6. 1·6 in. 7. 1 ft. 8. $a^2+2ab+b^2$. 9. $ax+ay+az+bx+by+bz$.
12. 8 sq. mi.; $\frac{1}{4}$ sq. in. 13. 0·6 sq. mi.; 20 sq. in.
14. 4·4 ac. 15. 21, 3, sq. in. 16. 7, 11, sq. in. 17. 24 sq. in.
18. 21 sq. in. 19. $17\frac{1}{2}$ sq. in. 20. $4\frac{1}{2}$, $23\frac{1}{2}$, $13\frac{1}{2}$, sq. in.
21. 3900 sq. yd. 22. 4 units. 23. 5 units. 24. 5·5 units.
25. 10 units. 26. 11 units. 27. 5 cm. 28. 3 in.
29. $3x$, $5x$, 30, sq. cm.; 3·75. 30. 24, 12, 36, sq. in.
31. $bx+ay=ab$. 32. 48 in. to the mile; $\frac{1}{2}$ in.
33. 12·57 sq. in.; 3·14 : 1. 34. 39 units.

**Page 244** ORAL EXAMPLES

6. (iv) 15, 7·5, sq. in. 7. 24 sq. in., 12 sq. in., 3 in., 3·2 in.
8. 43·5 sq. cm. 9. 15·0 sq. cm.

**Page 248** ORAL EXAMPLE

84 sq. in.; 11·2, 12, $12\frac{12}{13}$, in.

**Page 248** EXERCISE 50

1. 3·63, 4·53, cm.; 18·1 sq. cm. 2. 20 sq. cm.; 4, 4·8, in.
3. 17·5 sq. in.; 10 cm., 4·8 in. 4. 15, 15, 5, sq. cm.; $\frac{2}{7}$; $\triangle$PAC.
5. 96 sq. in., 9·6 in. 6. 24, 48, sq. in., 4·8 in. 7. 12 sq. in., 2·4 in.
8. 20, 17·5, 7·5, sq. in.; $3\frac{1}{3}$, $4\frac{3}{8}$, $1\frac{7}{8}$, in.; 45 sq. in.
9. 26 sq. in. 10. 25·5, 58·5, sq. in. 11. 9 sq. cm.; 4·5, 3·6 cm.
12. 4, 4·8, cm.; 53° 8′. 13. 3·6, 4·5, cm. 14. 10 in.
15. 4·8 in. 16. 7·5 sq. in. 17. 4 in. 18. 15 sq. in.
19. 5, 10, cm. 20. 5·4 sq. in. 21. 126 sq. in., 12 in.

22. 84 sq. in., 8 in. 23. 4·5, 4, cm. 24. 6, 4, in.
25. 4, 6, 8, in. 26. 6, 8, 4, sq. cm.
27. 48, 19·2, $8\frac{4}{13}$, $38\frac{10}{13}$, $17\frac{47}{65}$, sq. in.

**Page 264** EXERCISE 52

1. $37\frac{3}{4}°$ or $142\frac{1}{4}°$. 2. 5·74 cm. 3. 5·69 cm. 4. $47\frac{1}{2}°$.
5. 51°. 6. $38\frac{1}{2}°$. 7. 6 cm.; 41·6 sq. cm. 8. 5·68 cm.
9. 2·06, 4·03, in. 11. 18 sq. cm. 12. 29·1 sq. cm. 18. 4·8 cm.
19. 1·56 in. 20. 2·90 in. 21. 46°. 22. 5 cm.

**Page 271** ORAL EXAMPLES

(i) No; (ii) Yes.

**Page 272** ORAL EXAMPLES

1. 16, 16, 9, 9, sq. in.; 3·2, 1·8, 3, in.
2. 36, 36, 64, 64, sq. in.; 3·6, 6·4, 8, in.
3. 144, 25, 144, 25, sq. in.; 13 in. 4. 5 sq. in.; 2·24 in.
5. 3·16. 6. 2·65. 7. 25 sq. in.; $8\frac{1}{3}$, $5\frac{1}{3}$, $6\frac{2}{3}$, in.
8. $\frac{1}{2}\sqrt{3}=0{\cdot}866$, $\sqrt{3}=1{\cdot}732$.

**Page 273** EXERCISE 53

1. 17. 2. 13. 3. 3·61. 4. 20. 5. 3·32. 6. 2·83.
7. 3, 5·20. 8. 8, 6·93. 9. 6·46 in. 10. 16 ft.
11. 19·7 ft. 12. 15·8 mi. 13. 8·60 in. 14. 5·83 cm.
15. 162 mi. 16. 10 cm. 17. 6 in. 18. 17 cm.; 114 sq. cm.
19. 8·66 in., 45·6 sq. in. 20. 13 in. 21. 40 sq. cm.
22. 4·77 in. 23. 60 sq. in., $9\frac{3}{13}$ in. 24. 48 sq. in., 9·6 in.
25. 4·47 cm. 26. 15,000 yd. 27. 3·61 in. 28. 3·32 in.
29. 5 in. 30. 3·61 units. 31. 5 units. 32. 210 sq. cm.
33. 204 sq. in. 34. Obtuse-angled. 35. Acute-angled.
37. 13 in. 38. $6\frac{1}{4}$ in. 39. $8\frac{1}{3}$ in. 40. 3·57 in.
41. 7 in 42. $3\frac{1}{6}$, 3·87, cm. 43. 8 ft. 10 in.
44. $x^2(x^2+y^2)$, $y^2(x^2+y^2)$, sq. cm.
45. area $=abc(a+b)$; 70, 24, 74; 42, 40, 58; 112, 15, 113; are sides of equivalent right-angled triangles.
46. 16 ft. 47. 9 in. 48. 9 ft.

**Page 277** ORAL EXAMPLES

1. 485, 566; 23·8 ft. 2. 9·11 cm. 4. 119 sq. in.; 10·5 in.
6. 27 sq. cm.; 12 sq. cm.; 7·21 cm. 8. 8 cm.

**Page 280** EXERCISE 54

1. 26·8 ft. 2. 8·66 in. 3. 8 ft. 9 in. 4. 4·24 in.
5. 18 ft. 6. $1\frac{5}{8}$ sq. ft. 7. 260 sq. ft. 8. 13 cm.
9. 9·16 in. 10. 8·94 in. 11. 5 in. 12. 4 cm.; 5·83 cm.
13. 60 sq. in.; 11·7 in. 14. 12 cm. 15. 6·16 in. 16. 3 cm.
17. 8 in. 18. 8 in. 19. 2·24 in. 20. 9·90, 7·07, 8·77, in.
21. 7·34 in. 22. 2 in.

REVISION PAPERS, 35–50

**Page 288**

*Paper* 35. 1. $\frac{1}{2}(180-3x)$ degrees. 2. 2·66 in.
*Paper* 36. 1. 48°, 60° 72°; 45°, 60°, 75°; 40°, 60°, 80°.
3. $9\frac{1}{2}$ sq. in.

**Page 289**

*Paper* 37. 3. 3·54 in.

**Page 290**

*Paper* 39. 1. 36°. 3. 4 in.
*Paper* 40. 1. KC, BK. 3. 3·77 sq. in.
*Paper* 41. 3. $77\frac{11}{17}$, $33\frac{11}{13}$, in.; 2.

**Page 291**

*Paper* 42. 1. 15°. 3. 3, 4·5, sq. in.
*Paper* 43. 2. (i) 39 sq. in., 7·5 in., 10·4 in.; (ii) 7·79 sq. in.

**Page 292**

*Paper* 44. 2. $17\frac{1}{2}$ sq. in., $2\frac{1}{2}$ in.
*Paper* 45. 1. (i) 10·6 sq. in.; 4·31 in.
*Paper* 46. 2. 5 cm.

**Page 293**

*Paper* 47. 1. 12·5 cm. 4. 2·83 in.
*Paper* 48. 2. 2, $3\frac{5}{9}$, cm. 3. 8·25 in.; 2 in., 8·25 in.

**Page 294**

*Paper* 49. 1. (i) $pq(p^2-q^2)$ sq. in.; (ii) 17·3 sq. in., 4·33, 7·81, in.
4. 6·16 in.
*Paper* 50. 1. 10, 17·3, cm. 2. 7·5 sq. cm., 5·83 cm.

## PART II (*Section* 2)

**Page 296** ORAL EXAMPLES

1. 3 cm. 2. 9 in. 4. $9\frac{3}{8}$ cm.

**Page 297** EXERCISE 56

1. 13 cm. 2. 4·47 cm. 3. 11·5 cm. 4. Circle, radius 6 cm. 5. 8 cm. 6. 8·58, 0·583, cm. 7. 13·0 in. 8. 11·3 cm. 9. 8 in. 10. 5·83 cm. 11. $8d$ in. 12. 8·94 in. 13. 9 in. 14. $8\frac{1}{3}$ cm., 11·1 cm. 15. $7\frac{1}{24}$ in. 16. 3·46 cm. 17. 5 in. 19. 5·38 in. 20. 6·5 cm. 21. 4·8 in.; 3·6, 6·4, in. 22. $\frac{1}{2}$ in. 23. 2, $2\frac{1}{2}$, in.

**Page 317** EXERCISE 58

1. 55°. 2. 37°. 3. 65°. 4. 60°. 5. 107°. 6. 105°. 7. 72°. 8. 40°. 9. 128°. 10. 30°. 11. 54°, 99°. 12. $180 - \frac{1}{2}y$. 13. 110°. 14. 25°. 15. 70°. 17. 124°. 18. 105° or 5°. 19. 100°, 110°. 21. 54°. 23. 38°. 25. 76°, 98°, 132°, 124°, 110°. 26. 8 cm. 27. 5·29 cm. 28. 8·94, 4·47, cm. 29. 2·12 in. 30. 2·45 in.

**Page 335** EXERCISE 60

1. (i) No, (ii) yes. 2. (i) Yes, (ii) no. 3. 35°. 4. 45°. 5. 40°. 6. $\angle BDC = 50°$, $\angle DCB = 65°$. 8. 25°. 10. 60°, 70°, 50°. 13. 2 in.

**Page 343** EXERCISE 62

1. 18°. 2. 12°. 3. 105°. 4. 135°. 5. $\frac{1}{15}$. 6. $\frac{3}{4}$. 7. 3 : 2. 8. 35°. 9. $\frac{2}{9}$. 11. 3 : 2. 12. $53\frac{1}{3}$. 13. 60. 14. 50. 15. 10·5, 6·98, cm. 16. 2·9 cm.

**Page 344** EXERCISE 63

1. 3·146. 2. 25·1 in. 3. 22·0 cm. 4. 628 yd.
5. 1·7(5) in. 6. 1·4 cm. 7. 70 yd. 8. 0·95 ft.
9. 9·0 in. 10. 3·49 cm. 11. 16·8 cm. 12. 46°.
13. 4·3 cm. 14. 3·14. 15. 314 sq. in. 16. 50·3 sq. cm.
17. 154 sq. ft. 18. 14 in. 19. 3·5 cm. 20. 22·0 sq. in.
21. 5·89 sq. cm. 22. 4·57 sq. in. 23. 48·8 sq. in.
24. 45°, 105°, 30°. 25. 72°, 72°, 36°.
26. $7\frac{1}{2}$°, $22\frac{1}{2}$°, 150° or $127\frac{1}{2}$°, $22\frac{1}{2}$°, 30°. 27. 45°, 75°, 60°.
28. 46°, 37°. 29. 87°, 108°. 31. 3 : 1. 33. 16°. 34. $\frac{5}{9}$.
35. 77·4 sq. cm. 36. 2·5 cm. 37. 7 in. 38. 5 in.

**Page 355** EXERCISE 65

1. 65°, 50°. 2. 35°. 3. 8 cm. 4. 13 cm. 5. 110°.
6. 68°. 7. 72°. 8. 117°. 9. 52°, 38°. 10. 42°.
11. 155°. 12. 3 in. 13. 8 cm. 14. 60°, 65°, 55°.
15. 128°; 44°, 52°, 84°. 16. 5 cm. 17. 2·5 in. 18. 120°.
19. 77°, 90°, 103°, 90°. 20. 8, 2, cm. 21. 6, 1, in.
22. 12 cm. 23. 17 cm. 24. $2\frac{1}{6}$ in. 26. 16 in.

**Page 366** ORAL EXAMPLES

6. 1·56 in.

**Page 367** EXERCISE 67

1. 56°. 2. 36°. 3. 68°, 65°, 47°. 4. 65°, 75°, 40°.
6. 58°, 64°. 7. 100°. 8. 63°, 54°, 63°. 9. 94°, 8°.
10. 54° or 99°. 11. 79°, 114°, 101°, 66°. 12. $2x+y=90$.
13. $\angle$ABP = $\angle$TPK = 75°. 14. 72°, 65°. 15. 53°, 28°.

**Page 375** ORAL EXAMPLES

4. 5·5, 6·5, 7, cm. 5. 1·8, 1·4, 0·8, in. 6. 2, 1·5, 1·2, in.
9. 2·1 cm. 10. 1·75, 5·25, cm.

**Page 377** EXERCISE 69

1. 3 cm. 2. 6 cm. 3. $7\frac{1}{2}$, $4\frac{1}{2}$, in. 4. 2·5, 1·5, 4·5, in.
5. 4·5, 3·5, 2·5, cm. 6. 8, 4, 3, in. 7. 5·3, 3·6, 4·5, cm.
12. $1\frac{7}{8}$ in. 13. 2 cm. 14. 32, 8, cm. 15. 1·5 cm.
16. 4·45, 11·125, in. 17. 15 cm. 18. $5-3\sqrt{2}$, $=0·76$, ft. 19. $2\frac{1}{6}$.

**Page 382** EXERCISE 71

1. 4 cm. 9. 12 cm. 10. 19·1, 12, cm. 11. 5·45 cm.
12. 5·59 cm. 14. $\sqrt{\{d^2-(a+b)^2\}}$, $\sqrt{\{d^2-(a-b)^2\}}$, in.
16. 7, 1, cm.

**Page 394** EXERCISE 72

1. 3·49 cm. 3. 6·93 cm. 4. 6·93 cm. 6. 3·11 cm.
9. $b < \frac{1}{2}(d-a)$. 14. 0·65, 5·81, 1·94, 1·16, cm. 15. 4·61 cm.
16. Yes. 17. 1·46 cm. 18. 2·67 cm. 30. 2·66 cm.
31. 1·56 in. 32. 5·80 cm. 33. 8·13 cm. 34. 5·60, 2·14, cm.
35. 6·07, 4·02, cm. 38. 4·16 cm. 40. 3·2 cm. 41. 1·80 cm.

**Page 401** EXERCISE 73

5. 20°.

**Page 404** EXERCISE 74

3. 6·65 or 1·35, cm. 4. 2·63 cm.; yes. 6. Yes. 8. 4·47 cm.
15. 3·20 cm. 25. 5·87, 2·23, cm.

**Page 414** ORAL EXAMPLES

7. 2·299; 4·551. 8. 4·081; 11·46. 9. 82° 49′; 41° 24′.
10. 101° 32′; 34° 3′.

**Page 415** EXERCISE 76

1. Obtuse. 2. Acute. 3. Obtuse. 4. Obtuse. 5. 19, 10, in.
7. 5·85, 6·84, in.; 34·2 sq. in. 8. 1·375, 2·67, cm.; 5·33 sq. cm.
9. 11, 6·93, in.; 34·6 sq. in. 10. 3, 8·48(5), cm.; 42·4 sq. cm.
11. 1·2 cm. 13. 6·63 in. 14. $9\frac{5}{8}$ ft. 15. 4 cm.
16. 6·5, 8·5, in. 17. 5·45(5), 6·52, 7·97, cm. 18. 10 in.
19. 12·7 cm. 20. 9·16(5) cm. 21. 11, 9, cm. 22. 12·2 cm.
25. $(3a+b)$ in. 26. $4\frac{4}{5}$ in. 28. $2\frac{1}{2}$ in.

**Page 423** EXERCISE 78

1. 4 in. 2. 10 cm. 3. 12 cm. 4. 4 in. 5. 24 sq. in.
6. 13 in. 7. 6 cm. 8. $7\frac{1}{2}$ in. 9. 56 sq. cm. 10. 8 in.
11. 3·2 cm. 12. 1 cm. 16. 8, 7, 12, cm. 17. 5, 3, cm.
18. 6, 5·6, 15, cm. 19. 2, 4, 7·8, cm. 20. 2·25 cm.
21. 20 ft. 22. $10\frac{2}{3}$, $8\frac{1}{3}$, cm. 23. 3960 mi. 24. 3·54, 6·52, in.
25. $2\frac{1}{2}$ ft. 26. 34 in. 27. 6·5 cm. 28. 3, 30, mi.

**Page 435** EXERCISE 80

1. 6·32 cm. 2. 5·57 cm. 3. 1·97 in. 4. 8·06 cm. 5. 4·58 cm.
6. 8, 3. 7. 7·37, 1·63. 10. $AB < 2BC$.

**Page 438** EXERCISE 82

2. $\angle BRC = y$; $\angle CQD = z$; $\angle CDQ = \angle BDR = \angle BAC$.
3. $\angle BDC$. 4. $m+n$. 6. No, $BR \parallel CQ$. 7. No, $PD \parallel CQ$.
8. $\angle RBC = 3\angle RCB$. 9. $3\angle CDP - \angle BDP = 180°$.
10. $PQ \parallel BA$; $PR \parallel CA$; $CQ = QA$.
11. $\angle BDA - \frac{1}{2}\angle BCA = 90°$. 12. $\frac{1}{2}(\angle APB - \angle ABP)$.
24. YS. 37. Supplementary. 38. 99°, 24°, 33°.
39. $AB \parallel DC$; $60° - \frac{1}{3}\theta$; $120° - \frac{2}{3}\theta$; $\frac{4}{3}\theta - 60°$. 40. AG.
41. ABCD is rectangle.

REVISION PAPERS, 51–80

**Page 441**

*Paper* 51. 2. 55·2 sq. in. 3. 14 cm.
*Paper* 52. 3. 4 in.
*Paper* 53. 2. 2·9 cm. 3. 162°.

**Page 442**

*Paper* 54. 2. 4·57 cm.
*Paper* 55. 2. 12 sq. cm.; 4·8 cm. 3. 47°.

**Page 443**

*Paper* 56. 2. 3·25 cm.
*Paper* 57. 2. 29·1 sq. cm. 3. 37°

**Page 444**

*Paper* 58. 3. 3·6 in.
*Paper* 59. 2. 17·14 in.

**Page 445**

*Paper* 61. 2. 22°
*Paper* 62. 2. 55°, 40°

**Page 446**

*Paper* 63. 2. 5, 7, in.

**Page 447**

*Paper* 65. 1. 13 in. 3. 36°.
*Paper* 66. 2. $h+\frac{a^2}{4h}$. 3. 60°
*Paper* 67. 2. 15°.

**Page 448**

*Paper* 68. 3. 5·66, 8·48(5), cm.
*Paper* 69. 2. 51°.

**Page 449**

*Paper* 72. 2. $(90-2x)$ degrees.

**Page 450**

*Paper* 73. 3. 43·2 in.
*Paper* 74. 3. 5 in.
*Paper* 75. 4. 8 in.

**Page 451**

*Paper* 76. 3. 18·75 cm.
*Paper* 77. 1. 3. 2. 6·43 cm. 4. 12 cm.
*Paper* 78. 3. 7 cm.

**Page 452**

*Paper* 79. 1. (4, 3); (6, 5).
*Paper* 80. 3. 14, 16, cm. 4. 12·65, 6·92, in.

## *PART III*

**Page 455** EXERCISE 83

1. $3:8$. 2. $9:4$. 3. $2:3$. 4. $7\frac{1}{2}$. 5. $12\frac{1}{2}$.
6. $\pm 12$. 7. 10. 8. $p:x=y:q$. 9. $\mathrm{XA}:\mathrm{XT}=\mathrm{XT}:\mathrm{XB}$.
10. $\mathrm{AB}:\mathrm{PQ}=\mathrm{QR}:\mathrm{BC}$. 11. 3·2 in. 12. 14 in. 13. 2·4 in.
14. 14·1, 14·7, cm. 16. $16:1$. 17. $2:5$ externally, $1:2$ internally.
18. $\dfrac{2bc}{c+d}, \dfrac{2bc}{c\sim d}$, in. 19. $(x\sim y):2(x+y)$.
20. $d:c$, $b:d$; $a:c$. 21. $(c+d):d$; $c:(c+d)$.
25. $a+c+e$. 26. $d-f$. 27. $2b+7d-5f$.
28. $1:11$; $4:7$; $5:3$; $1:1$, $1:11$; 1·6 in.
29. $p:(p+q+r+s+t)$; $(q+r+s):(r+s+t)$;
$(p+q):(p+q+r+s+t)$, $(r+s):t$; $\dfrac{x(p+q+r+s)}{q+r}$ in.
30. $1:(n-1)$. 31. $2xy:(x^2\sim y^2)$; $(x\sim y):(x+y)$.
32. AY; XY; BD.

**Page 457** ORAL EXAMPLES

1. $3:8$, $3:8$. 2. $7:11$, $7:11$.
3. 14, 16; $\frac{7}{8}, \frac{7}{8}$; $\frac{7}{15}, \frac{7}{15}$. 5. $\dfrac{p}{p+q}, \dfrac{p}{p+q}$.

**Page 460** EXERCISE 84

7. 2·1 cm. 8. 9·6 cm. 9. 6, 10·5, cm. 13. 12, 10, cm.
14. $4\frac{1}{6}$ in. 15. 9, 8, cm. 16. 25, 12·8, cm. 17. $6\frac{2}{3}$, 9, cm.
18. 20, 15, cm. 19. 1·6 in. 20. 5·2 cm. 21. 2·8, 3·5, in.
25. $5:1$; $2:1$.

**Page 470** EXERCISE 86

1. $2\frac{1}{4}$, $11\frac{1}{4}$, in. 2. 3, 15, cm. 3. 8 in. 4. 3·35 in.
6. $\dfrac{ac}{b+c}$ units. 7. 12 in. 8. 9 sq. cm. 10. 1 in.
11. $9\frac{3}{4}$ sq. in. 12. 3 sq. in. 13. $(b+c):a$. 14. $3\frac{3}{7}$ cm. 15. 3 : 2.

**Page 477** EXERCISE 87

18. 14·4 cm.

**Page 480** EXERCISE 88

7. 7·5 cm. 8. 7·2 cm. 9. 4·2. 10. $4\frac{5}{7}$, $6\frac{2}{7}$.
11. 2·89. 12. 2·63. 13. 10·26. 16. 2·88 cm.

**Page 483** ORAL EXAMPLES

4. (i) $\triangle$s ABC, MNL, RPQ; (ii) 7·2, 6, 15, 22·5;
(iii) $\triangle$s ABC, FED. 6. (i) $\triangle$s AKC, DKB; (ii) 6.
8. $\triangle$s ABC, AED. 9. $\triangle$s PAB, PCD; $\triangle$s KAD, KCB.

**Page 485** EXERCISE 89

1. $3\frac{1}{3}$, 2, in. 2. 3·6, 2·4, in. 3. 6, $6\frac{2}{3}$, in. 4. 3·6, 3·6, in.
5. 1 : 4. 6. 3 : 2. 7. 3, $1\frac{2}{3}$, cm. 8. 12, 13·5, cm.
9. 2, 2·5, cm. 10. 8 cm. 11. 120 ft. 12. 4 ft.
13. 14·4 in. 14. 66 ft. 15. 5 cm. 17. 22·5 sq. in.
18. $8\frac{1}{3}$ sq. in. 19. 7·2 in. 20. 12 in. 21. 6, 6, cm.
22. 10, 7·5, 7·5, cm. 23. 10, 12, 5·5, 14, cm. 24. 4, 2, cm.
25. $2\frac{1}{3}$ in.; $7x+5y=35$. 26. 2·5, 20, in. 28. 6 cm.
29. 3·5, 11, in. 30. 2·4 in. 31. $\frac{3}{4}$, $1\frac{1}{8}$ in. 32. $6\frac{1}{2}$ in.

EXERCISE 89 (*continued*)

33. $2\frac{2}{3}$, $1\frac{1}{3}$, in. 34. $5\frac{1}{3}$, $3\frac{5}{9}$, in. 35. 18, 8, cm.
36. 6, 11, in. 37. $6\frac{2}{3}$ ft. 38. $8{\cdot}6 \times 10^5$ mi.; $2{\cdot}3 \times 10^5$ mi.
39. 12·8, 5, ft. 40. $8\frac{1}{3}$, $4\frac{1}{2}$, in. 41. 4 in. 43. 2·9 in.
44. $3\frac{3}{11}$ in. 45. $3\frac{1}{8}$ cm. 46. 15 ft. 47. 7 in.
51. $\dfrac{fx}{u-f}$.

**Page 498** EXERCISE 90

22. $\frac{3}{2}$, $\frac{3}{5}$. 23. $\frac{3}{4}$, 3.

**Page 502** ORAL EXAMPLES

3. 11 cm. 4. 3 cm. 5. 5 cm.

**Page 503** EXERCISE 92

1. 10·5 cm. 2. 3·5, 6·75, cm. 3. 6 cm. 4. 18 cm.
5. 2·31 in. 6. $21\frac{9}{11}$ in. 7. 6 cm. 8. $1\frac{12}{13}$ in.
9. 10 cm. 10. $4\frac{1}{2}$, $6\frac{1}{4}$, in.

**Page 510** EXERCISE 93

1. 7·07, 13·04, cm. 4. 9 cm. 5. $2\frac{1}{3}$, $4\frac{1}{6}$, in. 7. 4 : 5.
8. $\dfrac{p^2r}{q^2-p^2}$, $\dfrac{pqr}{q^2-p^2}$. 9. 0·707 in. 10. 13 in. 11. $5\frac{1}{2}$ in.

**Page 515** EXERCISE 95

1. 6·32 in. 2. 6·16 cm. 3. 5·57. 4. 5·29 cm. 5. 2·16 in.
6. 4·76 cm. 7. 3·95 cm. 8. 7·36 (or −1·36). 9. 5·00 cm.
10. 5·25 cm. 11. 5·5 cm. 12. 1·03 in.

**Page 516** ORAL EXAMPLES

1. 9; 25; 3:5, 9:25. 2. 5·6 cm.; 100:49 3. 25:16.

**Page 518** EXERCISE 96

1. 4:9. 2. 15:1. 3. 16:9. 4. 3:1.
5. 100, 36, 225. 6. 1:8; 1:3. 7. 1:16; 1:15.
8. 5:3; 25:9. 9. 12 in. 10. 4·2 cm. 11. 12 sq. ft.
12. 40 ac. 13. $101\frac{1}{4}$ sq. ft. 14. 9 ft. 15. £$5\frac{1}{3}$.
16. 1000. 17. $4\frac{17}{27}$ qt. 18. 512 lb. 19. 1·024 lb.
20. 6 in. 21. 2s. 3d. 22. $9\frac{13}{27}$ min. 23. 5 in.
24. 16:4:3:9. 25. 28:12:30:30:75.

**Page 528** EXERCISE 98

2. 3·63 cm. 3. 2·27 in. 4. 2·68 in. 5. 4·55 cm.
11. 10 cm. 16. 7·60 cm. 17. 6·60 cm. 18. 5·17 cm.
19. 4·57 cm.

REVISION PAPERS, 81–96

**Page 530**

*Paper* 81. 1. 8, 7. 3. 3:4; $5\frac{1}{4}$, $4\frac{1}{2}$, cm.

**Page 531**

*Paper* 82. 1. QA = 60 ft. 2 in., PA = 25 ft. $5\frac{1}{2}$ in.
3. $9\frac{1}{3}$, $\frac{1}{2}$, in.; 14, 40, cm.

**Page 532**

*Paper* 83. 1. 11 ft.
*Paper* 84. 3. 15, 9, $7\frac{1}{2}$, $22\frac{1}{2}$, cm.; $6\frac{3}{4}$ in.

**Page 533**

*Paper* 85. **3.** 2 : 3, 4 : 1, 1 : 3.

*Paper* 86. **1.** 7 cm., 6·49(5) sq. cm.

**3.** (i) 1 : 2, 3 : 1, 1 : 12; (ii) 5 : 4, $3\frac{1}{3}$ cm.

**Page 534**

*Paper* 87. **3.** 24, 10·8, cm.; 24·4 in.

*Paper* 88. **1.** 3·97, 4·5, 0·5, in. **3.** 9 cm.; 71 : 360.

**Page 535**

*Paper* 89. **2.** 4·47 in.

*Paper* 90. **1.** 11, 9, in. **2.** 9, $6\frac{1}{2}$, in.

**Page 536**

*Paper* 91. **2.** $5\frac{1}{3}$ sq. in. **3.** $3\frac{1}{3}$, $3\frac{2}{3}$.

*Paper* 92. **3.** $5\frac{1}{2}$, $2\frac{3}{11}$, in.; $1\frac{1}{2}$ in.

**Page 537**

*Paper* 93. **3.** 17, 7, 11.

*Paper* 94. **2.** 3, 2·22, cm. **3.** Circle, radius $2\frac{2}{3}$ cm.

**Page 538**

*Paper* 95. **2.** 7·49, 9·15, 1·66, in.; 4·8 cm.

# WORKS BY C. V. DURELL, M.A.

*ARITHMETIC* General Arithmetic for Schools
Arithmetic (with R. C. Fawdry)
A Concise Arithmetic (with R. C. Fawdry)

*ALGEBRA* A New Algebra for Schools
School Certificate Algebra
Advanced Algebra. Vol. I.
Vols. II and III (with A. Robson)
Elementary Algebra (with G. W. Palmer and R. M. Wright)
Matriculation Algebra (with G. W. Palmer)
Practical School Algebra
Graph Book (with A. W. Siddons)

*CALCULUS* Elementary Calculus (with A. Robson)
Vol. I, with appendix. Vol. II, with appendix
Higher Certificate Calculus (with A. Robson)

*GEOMETRY* A New Geometry for Schools. Stage 'A' and Stage 'B'. Issued separately or bound together
Elementary Geometry
Concise Geometry
A Shorter Geometry
Simplified Geometry (with C. O. Tuckey)
A First Geometry
School Geometry and Trigonometry. A unified course

*TRIGONOMETRY* Elementary Trigonometry (with R. M. Wright)
Advanced Trigonometry (with A. Robson)
Shorter Advanced Trigonometry (with A. Robson)
Matriculation Trigonometry
Stage 'A' Trigonometry. With Tables

*MECHANICS* A School Mechanics. In three parts

*GENERAL MATHEMATICS* Certificate Mathematics. In six volumes
General Mathematics. In four volumes. Also a supplementary volume
Revision Course in General Mathematics

*TABLES* Mathematical Tables

NEW
GEOMETRY
FOR
SCHOOLS

DURELL

*STAGE B*
PARTS 1,2 & 3

BELL